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Applying the Concept of Granger Causality to Detect Nonlinear Autodependencies in Time Series

1. Introduction

Granger causality is one of the most important concepts in the analysis of dependencies between economic processes. In econometrics, this widely known idea is usually applied to linear relationships, represented by VAR models. Nonlinear Granger causality may be identified using the Hiemstra and Jones test (1994).

The aim of this paper is to show how the Hiemstra and Jones test may be used for a different purpose, *i.e.* to detect nonlinear autodependencies in a single time series. This concept has been applied to some generated examples and financial data.

2. Testing for Nonlinear Granger Causality

The definition of Granger causality between two stationary time series is formulated in terms of conditional probability distributions (Granger, 1969). It says that X_t does not strictly Granger causes Y_t, if :

$$F\left(Y_t \,\middle|\, (X_{t-lx},...,X_{t-1};Y_{t-ly},...,Y_{t-1})\right)=F\left(Y_t \,\middle|\, (Y_{t-ly},...,Y_{t-1})\right) \qquad (1)$$

for each lags $lx, ly \geq 1$, where F denotes CDF. When the equality in Equation (1) does not hold we say that X_t strictly Granger cause Y_t.

Testing the Granger causality consists in verification of the null hypothesis that X does not strictly Granger cause Y. In practice, Equation (1) is not easy to apply to, therefore some more operational procedures are developed. For example, the Granger causality often becomes restricted to the linear framework, where VAR models for investigated data are analyzed.

Baek and Brock (1992) introduced an operational method of testing for nonlinear Granger causality. It is based on correlation integral $C_W(\varepsilon)$, which is the probability of finding two independent realizations of the vector W at a distance smaller than or equal to ε:

$$C_W(\varepsilon) = P\{\|W_1 - W_2\| \le \varepsilon\} = \int\int I(s_1, s_2, \varepsilon) f_W(s_1) f_W(s_2) ds_2 ds_1, \quad (2)$$

where W_1, W_2 are independent realizations of W, the integrals are taken over the sample of W, $\| \ \|$ is the supremum norm and $I(s_1, s_2, \varepsilon)$ denotes an indicator function: $I(s_1, s_2, \varepsilon) = \begin{cases} 1; & \|s_1 - s_2\| \le \varepsilon \\ 0; & \|s_1 - s_2\| > \varepsilon \end{cases}$.

For fixed lags $lx, ly \ge 1$, denote the lag vectors of X_t and Y_t, respectively by X^t_{t-lx} and Y^t_{t-ly}, *i.e.* $X^t_{t-lx} = (X_{t-lx}, \dots, X_t)$ and $Y^t_{t-ly} = (Y_{t-ly}, \dots, Y_t)$. Baek and Brock (1992) redefined the idea of Granger nonlinear causality. According to their definition X_t does not nonlinearly Granger cause Y_t, if:

$$\begin{aligned} &P\left\{\|Y_t - Y_s\| < \varepsilon \,\middle|\, \|Y^{t-1}_{t-ly} - Y^{s-1}_{s-ly}\| < \varepsilon, \|X^{t-1}_{t-lx} - X^{s-1}_{s-lx}\| < \varepsilon\right\} = \\ &= P\left\{\|Y_t - Y_s\| < \varepsilon \,\middle|\, \|Y^{t-1}_{t-ly} - Y^{s-1}_{s-ly}\| < \varepsilon\right\}. \end{aligned} \quad (3)$$

When the equality in Equation (3) does not hold, then knowledge of past X values helps to predict current and future Y values. This interpretation of Granger causality plays a crucial role in the concept presented in Section 3.

Let $C1$, $C2$, $C3$ and $C4$ denote the following correlation integrals:

$$C1 = P\left\{\|Y^t_{t-ly} - Y^s_{s-ly}\| < \varepsilon, \|X^{t-1}_{t-lx} - X^{s-1}_{s-lx}\| < \varepsilon\right\}, \quad (4a)$$

$$C2 = P\left\{\|Y^{t-1}_{t-ly} - Y^{s-1}_{s-ly}\| < \varepsilon, \|X^{t-1}_{t-lx} - X^{s-1}_{s-lx}\| < \varepsilon\right\}, \quad (4b)$$

$$C3 = P\left\{\|Y^t_{t-ly} - Y^s_{s-ly}\| < \varepsilon\right\}, \quad (4c)$$

$$C4 = P\left\{\|Y^{t-1}_{t-ly} - Y^{s-1}_{s-ly}\| < \varepsilon\right\}. \quad (4d)$$

It can be proved (e.g. Osińska and Orzeszko, 2006) that:

$$P\left\{\|Y_t - Y_s\| < \varepsilon \,\middle|\, \|Y^{t-1}_{t-ly} - Y^{s-1}_{s-ly}\| < \varepsilon, \|X^{t-1}_{t-lx} - X^{s-1}_{s-lx}\| < \varepsilon\right\} = \frac{C1}{C2}, \quad (5)$$

and $$P\left\{\|Y_t - Y_s\| < \varepsilon \,\middle|\, \|Y^{t-1}_{t-ly} - Y^{t-1}_{s-ly}\| < \varepsilon\right\} = \frac{C3}{C4}. \quad (6)$$

Thus, the null hypothesis of Granger noncausality given by (3) is equivalent to the equation $\frac{C1}{C2}=\frac{C3}{C4}$. (7)

Let consider two time series – (x_t) and (y_t), $t=1,2,...,T$, generated by strictly stationary stochastic processes X_t and Y_t. To verify Equation (7), the estimators of the correlation integrals $C1$, $C2$, $C3$ and $C4$ need to be calculated:

$$C1(n)=\frac{2}{n(n-1)}\sum\sum\nolimits_{t<s} I(y_{t-ly}^{t}, y_{s-ly}^{s}, \varepsilon) I(x_{t-lx}^{t-1}, x_{s-lx}^{s-1}, \varepsilon), \quad (8a)$$

$$C2(n)=\frac{2}{n(n-1)}\sum\sum\nolimits_{t<s} I(y_{t-ly}^{t-1}, y_{s-ly}^{s-1}, \varepsilon) I(x_{t-lx}^{t-1}, x_{s-lx}^{s-1}, \varepsilon), \quad (8b)$$

$$C3(n)=\frac{2}{n(n-1)}\sum\sum\nolimits_{t<s} I(y_{t-ly}^{t}, y_{s-ly}^{s}, \varepsilon), \quad (8c)$$

$$C4(n)=\frac{2}{n(n-1)}\sum\sum\nolimits_{t<s} I(y_{t-ly}^{t-1}, y_{s-ly}^{s-1}, \varepsilon), \quad (8d)$$

where $t,s=\max(lx,ly)+1,...,T$ and $n=T-\max(lx,ly)$.

According to the Hiemstra and Jones testing procedure (H–J hereafter), for given values of $lx,ly\geq 1$ and $\varepsilon>0$, under the assumptions that X_t and Y_t are strictly stationary, weakly dependent and satisfy the mixing conditions of Denker and Keller (1983), if X_t does not strictly Granger cause Y_t then:

$$TVAL \overset{df}{=} \frac{\sqrt{n}}{\sigma(lx,ly,\varepsilon)}\left(\frac{C1(n)}{C2(n)}-\frac{C3(n)}{C4(n)}\right)\sim N(0,1), \quad (9)$$

where the definition and the estimator of $\sigma(lx,ly,\varepsilon)$ are given in the appendix of Hiemstra and Jones (1994).

It should be emphasized that the Hiemstra and Jones test identifies dependencies of different types. Therefore, to examine nonlinear causality, first of all, the linear relation should be excluded.[1]

Moreover, Hiemstra and Jones recommend to analyze normalized time series and then, to consider the value of ε between 0.5 and 1.5.

[1] The test is usually applied to the estimated residual series from a VAR model.

3. Detection of Nonlinear Autodependencies Using the Hiemstra and Jones Test

In this paper it is proposed to apply the H–J test in a different way, *i.e.* to detect nonlinear autodependencies in a single time series. To this end, as the "causal process" one should take the past realisations of the investigated data. This allows to examine an existence of nonlinear autodependencies, which potentially allows to forecast the time series on a base of its past realisations.

Precisely, in this paper, such a procedure is realized in the two ways. For the each investigated time series, denoted by (a_t), two sets of time series are analysed:

***A*)** (y_t) – the investigated time series (*i.e.* $y_t = a_t$), (x_t) – the time series of its first lags (*i.e.* $x_t = a_{t-1}$),

***B*)** (y_t) – the time series of observations with even subscript (*i.e.* $y_t = a_{2t}$), (x_t) – the time series of observations with odd subscript (*i.e.* $x_t = a_{2t+1}$).[2]

In case *A*, the rejection of H_0 means that forecasts of a_t, based on observations $a_{t-1}, a_{t-2}, ..., a_{t-lx}$, will be improved, if we take into account also a_{t-lx-1}. Since $lx \geq 1$, it lets us identify autodependencies of at least second order. To find out first-order autodependencies the second set of the data, *i.e.* *B*, needs to be considered. In this case, a rejection of H_0 means that forecasts of a_{2t}, based only on observations $a_{2t-2}, a_{2t-4}, ..., a_{2t-2lx}$ will be improved, if we use $a_{2t-1}, a_{2t-2}, ..., a_{2t-2lx+1}, a_{2t-2lx}$. Particularly, when the value $lx = 1$ is considered, one can verify, if the observation a_{2t-1} influences a_{2t}. The main disadvantage of the variant *B* is that investigated time series (x_t) and (y_t) are twice shorter than the original data (a_t), which, obviously, decreases the power of the test.

Firstly, both procedures presented above, were applied to simulated data. From the logistic map $a_{t+1} = 4a_t(1 - a_t)$, for the initial state $a_0 = 0.7$, the chaotic time series of 1599 observations was generated. The value of ε =1.5 and the lags *lx*=*ly* equalled $1,2,...,5$, in turn, were considered in the test.

The results of this research, compared with the results obtained for the white noise time series, are summarized in Table 1. In each cell, the computed value of *TVAL* (see Eq. 9) is presented. The table header contains information, which

[2] To apply the H-J procedure, the DGP of (a_t) must satisfy the assumptions of this method (see section 2). In such a case, its subprocesses considered in *A* and *B*, fulfill these assumptions too.

set of time series (*A* or *B*) was analyzed. The symbols *, **, *** denote rejection of H_0 at 10%, 5% and 1% significant level, respectively[3].

As it can be seen, there is no evidence of any autodependencies in the white noise time series. The opposite, but simultaneously expected, result was obtained for the logistic map. For both variants – *A* and *B* a strong evidence of autodependencies was found. Since the further research showed no linear relations between the observations, we conclude that these dependencies are nonlinear. Moreover, the GARCH model was not able to capture them, which correctly indicates, that this time series was generated by the process of a different type.

Table 1. Results of H-J test for the white noise and the logistic map

lx=ly	White noise (*A*)	White noise (*B*)	Logistic map (*A*)		Logistic map (*B*)	
			series	GARCH(1,1)	series	GARCH(1,1)
1	-0.389	-0.532	**-13.689*** **	**-3.606*** **	**3.211*** **	**2.912*** **
2	0.312	0.569	**4.681*** **	-0.533	**3.658*** **	**3.289*** **
3	0.286	0.931	**-3.803*** **	0.827	**3.511*** **	**3.557*** **
4	0.453	0.751	1.411	0.012	**3.000*** **	**3.283*** **
5	0.102	-0.407	-1.023	-0.940	**2.241** **	**2.654*** **

Next the H–J test was applied to the Warsaw Stock Exchange indices from 2.01.2001-16.05.2007 (1600 observations). For the each index, the three time series were analysed: daily log returns, residuals from their ARMA and ARMA-GARCH models. Investigation of the residuals from the ARMA model gives information, if autodependencies are nonlinear. If so, the standardized residuals from the ARMA-GARCH model were analysed to verify if this class of processes can capture nonlinear dynamics of the investigated data. The results of this analysis are presented in Tables 2-11.

Table 2. Results of H-J test for the index WIG-BANKI

lx=ly	WIG-BANKI (*A*)			WIG-BANKI (*B*)		
	Returns	MA(1)	MA(1) GARCH(1,1)	Returns	MA(1)	MA(1)-GARCH(1,1)
1	-0.438	-0.542	**-3.949*** **	1.494	1.426	0.582
2	**3.011*** **	**3.027*** **	1.601	**2.462** **	**2.460** **	1.034
3	-0.137	0.182	**-1.861* **	**2.237** **	**2.156** **	0.411
4	**1.813* **	**1.658* **	0.528	**2.465** **	**2.423** **	0.659
5	1.136	1.190	-0.192	**3.068*** **	**2.988*** **	1.248

[3] According to the definition of causality (see Eq. 1), autodependencies are found, if, for at least one value of *lx*, the *TVAL* statistic falls in the critical region.

Table 3. Results of H-J test for the index WIG-BUDOW

lx=ly	WIG-BUDOW (*A*)			WIG-BUDOW (*B*)		
	Returns	AR(1)	AR(1)-GARCH(2,1)	Returns	AR(1)	AR(1)-GARCH(2,1)
1	**2.631*** **	**2.694*** **	0.412	**2.874*** **	**3.246*** **	0.639
2	**3.882*** **	**3.846*** **	**1.855* **	**3.959*** **	**4.292*** **	1.115
3	0.821	1.201	-0.239	**5.063*** **	**4.749*** **	1.479
4	**3.004*** **	**3.149*** **	1.227	**5.143*** **	**4.375*** **	1.442
5	**1.941* **	**1.980** **	0.493	**4.490*** **	**3.649*** **	0.785

Table 4. Results of H-J test for the index WIG-INFO

lx=ly	WIG-INFO (*A*)			WIG-INFO (*B*)		
	Returns	AR(1)	AR(1)-GARCH(2,1)	Returns	AR(1)	AR(1)-GARCH(2,1)
1	**4.187*** **	**4.230*** **	-0.262	**3.613*** **	**3.195*** **	-0.803
2	**4.069*** **	**4.128*** **	0.840	**4.127*** **	**4.093*** **	0.344
3	**4.782*** **	**4.694*** **	1.245	**4.281*** **	**3.916*** **	-0.569
4	**3.506*** **	**3.626*** **	-0.487	**3.738*** **	**4.197*** **	-0.268
5	**2.696*** **	**2.549** **	-0.430	**4.145*** **	**4.299*** **	-0.101

Table 5. Results of H-J test for the index MWIG40

lx=ly	MWIG40 (*A*)			MWIG40 (*B*)		
	Returns	ARMA(2,1)	ARMA(2,1)-GARCH(1,1)	Returns	ARMA(2,1)	ARMA(2,1)-GARCH(1,1)
1	**4.232*** **	**4.362*** **	0.469	**2.775*** **	**2.709*** **	0.369
2	**3.665*** **	**3.649*** **	-0.287	**4.149*** **	**4.296*** **	1.362
3	**3.528*** **	**3.696*** **	1.625	**5.405*** **	**5.538*** **	**2.026** **
4	**4.566*** **	**4.514*** **	0.573	**4.883*** **	**5.135*** **	1.040
5	**3.989*** **	**3.465*** **	0.373	**5.296*** **	**5.566*** **	1.416

Table 6. Results of H-J test for the index WIG-SPOZY

lx=ly	WIG-SPOZY (*A*)			WIG-SPOZY (*B*)		
	Returns	AR(2)	AR(2)-GARCH(1,1)	Returns	AR(2)	AR(2)-GARCH(1,1)
1	**4.126*** **	**3.949*** **	-0.796	**4.681*** **	**4.033*** **	0.514
2	**3.341*** **	**3.698*** **	**-1.687* **	**4.834*** **	**4.738*** **	0.143
3	**3.808*** **	**3.870*** **	0.054	**4.900*** **	**4.708*** **	0.289
4	**3.588*** **	**3.486*** **	0.750	**4.712*** **	**4.516*** **	-0.019
5	**2.879*** **	**3.103*** **	-0.565	**5.193*** **	**5.072*** **	-0.052

Table 7 Results of H-J test for the index SWIG80

lx=ly	SWIG80 (*A*)			SWIG80 (*B*)		
	Returns	ARMA(2,1)	ARMA(2,1) - GARCH(1,1)	Returns	ARMA(2,1)	ARMA(2,1) - GARCH(1,1)
1	**3.519*** **	**3.906*** **	1.180	**4.392*** **	**3.603*** **	**1.869* **
2	**3.706*** **	**2.667*** **	0.272	**4.213*** **	**3.529*** **	**1.803* **
3	**2.707*** **	**2.148** **	0.169	**4.084*** **	**3.701*** **	**1.908* **
4	**2.748*** **	**2.006** **	0.003	**4.369*** **	**3.770*** **	**1.968** **
5	**3.115*** **	**2.380** **	0.619	**3.217*** **	**3.176*** **	0.830

Table 8. Results of H-J test for the index TECHWIG

lx=ly	TECHWIG (*A*)			TECHWIG (*B*)		
	Returns	AR(1)	AR(1)-GARCH(1,1)	Returns	AR(1)	AR(1)-GARCH(1,1)
1	**4.299*** **	**4.196*** **	-0.294	**4.056*** **	**3.155*** **	-1.565
2	**4.393*** **	**4.804*** **	0.825	**4.616*** **	**4.459*** **	0.034
3	**5.322*** **	**5.390*** **	**1.894* **	**5.152*** **	**4.405*** **	-0.502
4	**4.018*** **	**4.158*** **	-0.223	**4.963*** **	**4.533*** **	-0.814
5	**3.091*** **	**3.093*** **	-0.520	**5.211*** **	**4.884*** **	-0.533

Table 9. Results of H-J test for the index WIG-TELKO

lx=ly	WIG-TELKO (*A*)		WIG-TELKO (*B*)	
	Returns	GARCH(1,1)	Returns	GARCH(1,1)
1	**3.797*** **	0.264	**3.558*** **	1.206
2	**3.273*** **	-1.121	**3.754*** **	0.490
3	**3.728*** **	1.294	**4.105*** **	0.905
4	**2.664*** **	0.155	**4.235*** **	0.453
5	**3.372*** **	0.844	**4.556*** **	0.643

Table 10. Results of H-J test for the index WIG

lx=ly	WIG (*A*)			WIG (*B*)		
	Returns	AR(1)	AR(1)-GARCH(1,1)	Returns	AR(1)	AR(1)-GARCH(1,1)
1	0.997	1.026	**-2.321** **	0.779	0.527	**-2.064** **
2	**3.401*** **	**3.368*** **	0.322	**1.791* **	**2.156** **	-1.258
3	**2.834*** **	**3.154*** **	0.301	**3.494*** **	**3.570*** **	-0.237
4	**4.018*** **	**4.025*** **	**2.266** **	**3.039*** **	**4.003*** **	0.228
5	**3.211*** **	**3.453*** **	0.338	**3.140*** **	**4.211*** **	0.620

Table 11. Results of H-J test for the index WIG20

lx=ly	WIG20 (*A*)			WIG20 (*B*)		
	Returns	MA(1)	MA(1)-GARCH(1,1)	Returns	MA(1)	MA(1)-GARCH(1,1)
1	0.625	0.725	**-2.949*** **	1.018	0.948	-1.176
2	**3.558*** **	**3.494*** **	0.658	**2.285** **	**2.246** **	-1.000
3	**2.816*** **	**3.034*** **	0.168	**3.287*** **	**3.164*** **	-0.304
4	**3.644*** **	**3.593*** **	1.621	**2.905*** **	**2.716*** **	-0.686
5	**3.140*** **	**3.146*** **	0.102	**2.851*** **	**2.790*** **	-0.723

The obtained results indicate that the evidence of autodependencies was found for the most investigated indices. The same conclusion may be drawn for the residuals from the ARMA models, which means that these autodependencies are nonlinear. Filtering by the ARMA-GARCH models made the modulus of the *TVAL* statistic smaller but in the most cases (WIG-BUDOW (*A*), MWIG40 (*B*), WIG-SPOZY (*A*), SWIG80 (*B*), TECHWIG (*A*), WIG (*A*)), these models were not able to capture the identified nonlinearities. Moreover, in some cases (WIG-BANKI (*A*), WIG (*B*), WIG20 (*A*)) filtering by the ARMA-

GARCH models increased the *TVAL* statistic, which may also confirm that the identified autodependencies are not driven by a GARCH process.

A presence of autodependencies makes an effective prediction of time series possible. Of course, the applied procedure provides no guidance regarding the source of the identified relations and so the method of forecasting. However, one should realize, that due to the variety of nonlinearities, an attempt to recover the generating mechanism seems to be futile. That is why, nonparametric methods of forecasting may be plausible for the data investigated in this paper (see e.g. Orzeszko, 2004). However, the problem of finding suitable techniques of prediction are beyond the scope of this paper.

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