DYNAMIC ECONOMETRIC MODELS
Vol. 8 – Nicolaus Copernicus University – Toruń – 2008

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Description of the Kurtosis of Distributions by Selected Models with Sign Function*

1. Introduction

Financial markets research indicated that financial series, such as stock returns, foreign exchange rates and others, exhibit leptokurtosis and volatility varying in time. An asymmetric reaction of the volatility to good or bad news (changes in returns) is observed. Hence, volatility tends to grow in reaction to bad news and fall in response to good news. In finance, this effect is called financial leverage. In models such as EGARCH, GJR and TARCH the negative correlation between returns and its volatility is taken into consideration (Brzeszczyński, Kelm, 2002; Doman, Doman, 2004; Fiszeder, 2001; Fornari, Mele, 1997). Fornari and Mele (1997) proposed a different way to take into account an asymmetry in comparison to GJR models. In their work, they have shown that the proposed models provide better interpretative results than the GJR models.

In the literature, non-linear dynamics of financial time series has generally been described by the class of GARCH models (Bollerslev, 1986; Engle, 1982). A different, alternative approach to the description of financial time series represents random coefficient autoregressive models (RCA) (Nicholls, Quinn, 1982). However, the RCA GARCH models with innovations from the normal distribution (Thavaneswaran, Appadoo, Samanta, 2005) can be treated as an alternative to GARCH models with innovations from the t-distribution (or similar) (Górka, 2007b). Including a sign function in the RCA model causes a change in the parameter value to depend on the sign of the previous observation. It is the similar in case of the RCA GARCH model.

* This work was financed from the Polish science budget resources in the years 2008-2010 as the research project N N111 434034.

The aim of this paper is to present the selected models with the sign function. The focus will be on properties of processes generated by selected models, especially on their kurtosis and its existence conditions.

2. Sign RCA Models

Random coefficient autoregressive models (RCA) are straightforward generalization of the constant coefficient autoregressive models. A full description of this class of models including their properties, estimation methods and some application, can be found in Nicholls and Quinn (1982).

The classical random coefficient autoregressive model of first order for stationary univariate time series can be written as:

$$y_t = (\alpha + \delta_t) y_{t-1} + \varepsilon_t , \tag{1}$$

where:

$$\begin{pmatrix} \delta_t \\ \varepsilon_t \end{pmatrix} \sim^{iid} \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_\delta^2 & 0 \\ 0 & \sigma_\varepsilon^2 \end{pmatrix} \right), \tag{2}$$

$$\alpha^2 + \sigma_\delta^2 < 1 . \tag{3}$$

Condition (3) is necessary and sufficient for the second-order stationarity of y_t. Conditions (2)-(3) ensure the strict stationarity of the process. The model (1), with appropriate assumptions, can take the form of: AR, STUR, RCA(1, p) (Górka, 2007a; Lee, 1998).

If conditions (2)-(3) are satisfied, the process (1) has the following properties (Appadoo, Thavaneswaran, Singh, 2006; Aue, 2004):

$$E(y_t) = 0 , \tag{4}$$

$$E(y_t^2) = \frac{\sigma_\varepsilon^2}{1 - \alpha^2 - \sigma_\delta^2} , \tag{5}$$

$$K = \frac{3\left[1 - (\alpha^2 + \sigma_\delta^2)^2\right]}{1 - (\alpha^4 + 6\alpha^2\sigma_\delta^2 + 3\sigma_\delta^4)} . \tag{6}$$

Hence, the process (1) has zero mean, constant unconditional variance and kurtosis. The constant unconditional variance of RCA(1) is bigger than the unconditional variance of AR(1). If $\sigma_\delta^2 = 0$, the value of kurtosis (6) reduces to 3 (similarly to AR(1) models). The necessary condition for the existence of the kurtosis (6) has the form: $\alpha^4 + 6\alpha^2\sigma_\delta^2 + 3\sigma_\delta^4 < 1$.

A stationary sign RCA models is given by (Thavaneswaran, Appadoo, 2006):

$$y_t = (\alpha + \delta_t + \Phi s_{t-1}) y_{t-1} + \varepsilon_t , \tag{7}$$

where the conditions (2)-(3) are satisfied, and

$$s_t = \begin{cases} 1 & \text{for } y_t > 0 \\ 0 & \text{for } y_t = 0 \\ -1 & \text{for } y_t < 0 \end{cases} . \tag{8}$$

If $\alpha + \delta_t > |\Phi|$, the negative value of Φ means, that the negative (positive) observation values at time $t-1$ correspond to a decrease (increase) of observation values at time t. In the case of stock returns it would suggest (for returns) that after a decrease of stock returns the decrease of stock returns occurs higher than expected, and in the case of the increase of stock returns the increase in stock returns occurs lower than expected.

If conditions (2)-(3) are satisfied, the process (7) has the following properties (Thavaneswaran, Appadoo, 2006):

$$E(y_t) = 0 , \tag{9}$$

$$E(y_t^2) = \frac{\sigma_\varepsilon^2}{1 - \alpha^2 - \sigma_\delta^2 - \Phi^2} , \tag{10}$$

$$K = \frac{3\left[1 - (\alpha^2 + \sigma_\delta^2 + \Phi^2)^2\right]}{1 - \left[\alpha^4 + \Phi^4 + 6\left[\alpha^2\sigma_\delta^2 + \Phi^2(\alpha^2 + \sigma_\delta^2)\right] + 3\sigma_\delta^4\right]} . \tag{11}$$

Therefore, the process described by equation (7) has zero mean, constant unconditional variance and kurtosis. The necessary condition for the existence of the kurtosis is $\alpha^4 + \Phi^4 + 6\left[\alpha^2\sigma_\delta^2 + \Phi^2(\alpha^2 + \sigma_\delta^2)\right] + 3\sigma_\delta^4 < 1$. (12)

When $\sigma_\delta^2 = 0$ and $\Phi = 0$, the kurtosis of (11) reverts to 3.

From the comparison of properties (4)-(6) of the RCA model with properties (9)-(11) of the sign RCA model we can see that introducing the sign function to the RCA model causes an increase of variance and kurtosis with relation to the variance and kurtosis obtained for the process described by the RCA model without sign function.

3. Sign GARCH Models

The general GARCH(p, q) model is described by the following equations:

$$y_t = \sigma_t \varepsilon_t , \tag{13}$$

$$\sigma_t^2 = \omega + \sum_{i=1}^{q} \alpha_i y_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2 , \tag{14}$$

where $\varepsilon_t \sim iid(0,1)$, $\omega > 0$, $\alpha_i \geq 0$ and $\beta_j \geq 0$.

Formulas for the theoretical kurtosis for processes generated by given GARCH models are presented in the literature (e.g. Doman, Doman, 2004). However, there is a lack of a general formula for the kurtosis of processes described by the equations (13)-(14). Below the general formula for the kurtosis of many GARCH models (i.e. with different assumptions about distributions) will be presented.

To write the general formula for the kurtosis of the GARCH model, without an assumption about the distribution type, the model (13)-(14) should be written as ARMA form. If $u_t = y_t^2 - \sigma_t^2$ is the martingale difference with variance $\text{var}(u_t) = \sigma_u^2$, the model (16)-(17) can be interpreted as ARMA(m, q) model for y_t^2 and can be written as:

$$y_t^2 = \omega + \sum_{i=1}^{m} (\alpha_i + \beta_i) y_{t-i}^2 - \sum_{j=1}^{p} \beta_j u_{t-j} + u_t \tag{15}$$

or $\phi(B) y_t^2 = \omega + \beta(B) u_t ,$ (16)

where $\phi(B) = 1 - \sum_{i=1}^{m} (\alpha_i + \beta_i) B^i = 1 - \sum_{i=1}^{m} \phi_i B^i$, $\beta(B) = 1 - \sum_{j=1}^{p} \beta_j B^j$, $m = \max\{p, q\}$, $\alpha_i = 0$ for $i > q$ and $\beta_i = 0$ for $j > p$.

The stationarity assumptions for y_t^2, which has an ARMA(m,q) representation, are the following (Thavaneswaran, Appadoo, Samanta, 2005):

(Z.1) All roots of the polynomial $\phi(B) = 0$ lie outside the unit circle.

(Z.2) $\sum_{i=0}^{\infty} \psi_i^2 < \infty$, where the ψ_i's are coefficients of the polynomial $\psi(B) = 1 + \sum_{i=1}^{\infty} \psi_i B^i$ satisfying the equation $\psi(B)\phi(B) = \beta(B)$.

These assumptions ensure that the u_t's are uncorrelated, have zero mean and finite variance and that the y_t^2 process is weakly stationary.

If the GARCH(p,q) model, described by (16), satisfies conditions (Z.1)-(Z.2) and has finite unconditional moment of fourth order, then the kurtosis K

of process y_t^2 specified by (16) can be written as (Thavaneswaran, Appadoo, Samanta, 2005):

$$K = \frac{E\left(\varepsilon_t^4\right)}{E\left(\varepsilon_t^4\right) - \left[E\left(\varepsilon_t^4\right) - 1\right]\sum_{i=0}^{\infty} \psi_i^2} . \tag{17}$$

The values of parameters for a given model are included in individual weights values. The formulas derived for kurtosis for given GARCH models with the normal distribution and the t-distribution can be found in Górka (2007b).

Fornari, Mele (1997) proposed the use of a sign function in the GARCH models. The general sign GARCH(p, q) model is given by:

$$y_t = \sigma_t \varepsilon_t \qquad y_t \middle| I_{t-1} \sim N\left(0, \sigma_t^2\right), \tag{18}$$

$$\sigma_t^2 = \omega + \sum_{i=1}^{q} \alpha_i y_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2 + \sum_{k=1}^{l} \Phi_k s_{t-k} , \tag{19}$$

where $\varepsilon_t \sim iid(0,1)$, $\omega > 0$, $\alpha_i \geq 0$, $\beta_j \geq 0$ and s_{t-1} is defined by formula (9) and $\left|\sum \Phi_k\right| \leq \omega$. The last condition ensures nonnegative values of $\left\{\sigma_t^2\right\}$.

The attempts to find the general formula for kurtosis of process, similarly to the GARCH model (13)-(14), have been made in the literature (Thavaneswaran, Appadoo, 2006). However, the general formula of kurtosis presented by Thavaneswaran, Appadoo (2006) does not give the same results as those obtained by author or by Fornari, Mele (1997). Discovering the general formula of kurtosis for the sign GARCH process, in my opinion, is possible, however, it needs further research.

Let, the sign ARCH(1) model be given by:

$$y_t = \sigma_t \varepsilon_t , \qquad \sigma_t^2 = \omega + \alpha_1 y_{t-1}^2 + \Phi s_{t-1} , \tag{20}$$

where $\varepsilon_t \sim N(0,1)$, $\omega > 0$, $\alpha_1 \geq 0$, $|\Phi| \leq \omega$, s_{t-1} is defined by formula (8).

Supposing $\Phi < 0$, negative (positive) observation values at time $t-1$ correspond to an increase (decrease) in conditional variance values at time t . Therefore, negative value of Φ denote negative correlation between volatility and returns.

Assuming the stationarity conditions of process y_t^2 specified by (20) are satisfied the moments are the following:

$$E(y_t) = 0 , \tag{21}$$

$$E\left(y_t^2\right) = \frac{\omega}{1-\alpha_1} , \tag{22}$$

$$K = \frac{3\Phi^2(1-\alpha_1)^2 + 3\omega^2(1-\alpha_1^2)}{\omega^2(1-3\alpha_1^2)}. \qquad (23)$$

Therefore, the sign function does not influence the unconditional mean and variance. However the unconditional kurtosis for the process, described by (20), increases about $\frac{3\Phi^2(1-\alpha_1)^2}{\omega^2(1-3\alpha_1^2)}$ in comparison with ARCH model (with innovations from the normal distribution). The conditional assumption of the existence of the kurtosis does not change too (Górka, 2007b), i.e. $\alpha_1^2 < \frac{1}{3}$.

In the same way, like for the sign ARCH model the kurtosis for the process described by the sign GARCH(1,1) model can be obtained. Let the model be as follows:

$$y_t = \sigma_t \varepsilon_t, \qquad \sigma_t^2 = \omega + \alpha_1 y_{t-1}^2 + \beta_1 \sigma_{t-1}^2 + \Phi s_{t-1}, \qquad (24)$$

where $\varepsilon_t \sim N(0,1)$, $\omega > 0$, $\alpha_1 \geq 0$, $\beta_1 \geq 0$, $|\Phi| \leq \omega$ and s_{t-1} is defined by formula (8).

Then, under the stationarity assumption for process y_t^2, we obtain:

$$E(y_t) = 0, \qquad (25)$$

$$E(y_t^2) = \frac{\omega}{1-\alpha_1-\beta_1}, \qquad (26)$$

$$K = \frac{3\omega^2(1-(\alpha_1+\beta_1)^2) + 3\Phi^2(1-(\alpha_1+\beta_1))^2}{(1-(\alpha_1+\beta_1)^2 - 2\alpha_1^2)\omega^2}. \qquad (27)$$

For the sign GARCH(1,1) model, like for the sign ARCH(1) model, the value of unconditional kurtosis increases in comparison with ordinary GARCH model (with innovations from the normal distribution). The unconditional mean and variance do not change. The conditional assumption of the existence of the kurtosis does not change too (Górka, 2007b), i. e. $(\alpha_1+\beta_1)^2 + 2\alpha_1^2 < 1$.

Summing up, the introduction of sign function into the GARCH model produces only the increase of kurtosis. If $\Phi = 0$, then formulas (23) and (33) of the kurtosis process are reduced to formulas of the kurtosis processes generated by appropriate GARCH models (Górka, 2007b).

4. Sign RCA GARCH Models

The RCA GARCH models, in presented form, were proposed by Thavaneswaran, Appadoo, Samanta (2005). In the RCA GARCH models, like in the AR model case, a random coefficient is introduced into the GARCH model (Górka, 2007b; Thavaneswaran, Appadoo, Samanta, 2005). When the sign function is added to the RCA GARCH model, then the sign RCA GARCH model is obtained. The sign function is obtained in different way than in the sign GARCH model case. The sign RCA ARCH(1) model has the form:

$$y_t = \sigma_t \varepsilon_t \,,\; \sigma_t^2 = \omega + (\alpha_1 + a_{t-1} + \Phi s_{t-1}) y_{t-1}^2 \,, \tag{28}$$

where $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$, $a_t \sim N(0, \sigma_a^2)$, and s_{t-1} is defined by formula (8).

A negative value of Φ denotes, that for negative (positive) values of observations at time $t-1$ an increase (decrease) of volatility occurs at time t. If $u_t = y_t^2 - \sigma_t^2$, then the equation for conditional variance is given by:

$$y_t^2 = \omega + (\alpha_1 + a_{t-1} + \Phi s_{t-1}) y_{t-1}^2 + u_t \,. \tag{29}$$

Therefore, the equation for conditional variance in the sign RCA ARCH(1) model can be interpreted as the sign RCA model for y_t^2. Under the stationarity assumption for the process y_t^2 we obtain (por. Thavaneswaran, Appadoo, 2006):

$$E(y_t) = 0 \,, \tag{30}$$

$$E(y_t^2) = \frac{\sigma_\varepsilon^2 \omega}{1 - \sigma_\varepsilon^2 \alpha_1} \,, \tag{31}$$

$$K = \frac{3(1 - \alpha_1^2 \sigma_\varepsilon^4)}{1 - 3\sigma_\varepsilon^2 (\alpha_1^2 + \sigma_a^2 + \Phi^2)} \,. \tag{32}$$

In this case, only the kurtosis was changed (it was increased). The necessary condition of the existence of the kurtosis of the sign RCA ARCH model is $\sigma_\varepsilon^2 (\alpha_1^2 + \sigma_a^2 + \Phi^2) < \frac{1}{3}$. It is worth noticing, that this condition, when compared to that of the existence of the kurtosis of process generated by RCA GARCH model, has been changed.

In the case when the parameter standing by y_{t-1}^2 in the GARCH(1,1) model is random, we have the sign RCA GARCH(1,1) model, that should be written as:

$$y_t = \sigma_t \varepsilon_t \,, \quad \sigma_t^2 = \omega + (\alpha_1 + a_{t-1} + \Phi s_{t-1}) y_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \,, \tag{33}$$

where $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$, $a_t \sim N(0, \sigma_a^2)$, and s_{t-1} is defined by formula (8).

Under the stationarity assumption for the process y_t^2 we obtain:

$$E(y_t) = 0 \,, \tag{34}$$

$$E(y_t^2) = \frac{\sigma_\varepsilon^2 \omega}{1 - \sigma_\varepsilon^2 \alpha_1 - \beta_1} \,, \tag{35}$$

$$K = \frac{3\left[1 - \left(\alpha_1 \sigma_\varepsilon^2 + \beta_1\right)^2\right]}{1 - 2\sigma_\varepsilon^2 \alpha_1 \beta_1 - 3\sigma_\varepsilon^4 \left(\alpha_1^2 + \sigma_a^2 + \Phi^2\right) - \beta_1^2}. \tag{36}$$

For the sign RCA GARCH model only kurtosis is increased. The necessary condition of the existence of the kurtosis for the process described by the sign RCA GARCH(1,1) is $2\sigma_\varepsilon^2 \alpha_1 \beta_1 + 3\sigma_\varepsilon^4 \left(\alpha_1^2 + \sigma_a^2 + \Phi^2\right) + \beta_1^2 < 1$.

In each of the presented cases, the introduction of a sign function caused the increase of the kurtosis of process. When $\Phi = 0$ formulas (32) and (36) for the kurtosis of process are reduced to formulas for the kurtosis of the process generated by appropriate RCA GARCH models (Górka, 2007b).

5. Summary

In this paper the selected models with a sign function are presented to describe the asymmetric behaviour of the conditional volatility (or, returns in the RCA model case) with respect to the occurrence of negative or positive shocks (information). The parameter of the sign function, for each presented model, causes the increase of kurtosis. The necessary conditions of the existence of the kurtosis of the process suggest the limited use of the normal distribution for innovations in presented models.

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