DYNAMIC ECONOMETRIC MODELS
Vol. 6 – Nicolaus Copernicus University – Toruń – 2004

Witold Orzeszko
Nicolaus Copernicus University in Toruń

How the Prediction Accuracy of Chaotic Time Series Depends on Methods of Determining the Parameters of Delay Vectors

1. Introduction

Unlike truly random processes, chaotic dynamics can be forecasted very precisely in a short run. In this paper, one of the methods applied to predicting chaotic dynamics – a local polynomial approximation, has been presented. A first step of this method is a reconstruction of system states by considering delay vectors. This procedure requires determining two parameters: an embedding dimension and a time delay. Examples of the methods developed for this purpose are false nearest neighbours and mutual information. The aim of this paper is to examine an adequacy of these techniques in application to forecast methods. In addition, an alternative method of determining the parameters of delay vectors is proposed.

2. The state space reconstruction

The main object of interest in the chaos theory is a dynamic system, which is formally defined as a pair (S, f), where $S \subset R^d$ is a set of system states and $f: S \to S$ is a map describing the dynamics of these states. Investigating a time series with methods from the chaos theory, one assumes, that it has been generated by a certain dynamic system (S, f). The first step of such an investigation is taken by reconstructing the state space *S*. Its purpose is to uncover information about the states of the unobserved generating system and their dynamics, from the time series. The most widely used method of state space reconstruction is currently the technique of delay coordinates, which consist in constructing delay vectors (so called *m*-histories) in a form of $\hat{x}_t^m = (x_t, x_{t-1\cdot lag}, ..., x_{t-(m-1)\cdot lag})$, where *m* is called an embedding dimension.

A theoretical base for the method of delays is Takens' theorem, which states that, for $m \geq 2d+1$ a system $(\hat{S}, F)$, where $\hat{S}$ is a set of *m*-histories and F is a map defining its dynamics, ie $F(\hat{x}_t^m) = \hat{x}_{t+1}^m$, may be used to investigate properties of the unknown system (S, f). For example, an estimation of the attractor's dimension and the Lyapunov exponents is possible. Moreover, a reconstructed system $(\hat{S}, F)$ may be used to forecast the original motion of the system (see Castagli et al. (1991)).

In the delay coordinates method, the values of an embedding dimension *m* and a time delay *lag* must be *a priori* established. The proper choice of these parameters' values is particularly important in the case of short and noisy time series (see Bask (1998), Castagli et al. (1991), Zeng et al. (1991)). The false nearest neighbours – FNN method may be used to determine a value of the embedding dimension (Kennel et al. (1992)). Its idea is to calculate an amount of, so called, false nearest neighbours as a function of the parameter *m*, in the following order:

-) for all *m*-histories $\hat{x}_{t1}^m$ and $\hat{x}_{t2}^m$ the norm $\left\| \hat{x}_{t1}^m - \hat{x}_{t2}^m \right\|$ is calculated, (1)

-) the coefficient $d_m = \left\| \hat{x}_{t1}^{m+1} - \hat{x}_{t2}^{m+1} \right\| / \left\| \hat{x}_{t1}^m - \hat{x}_{t2}^m \right\|$ is calculated. (2)

If d_m is bigger than a certain fixed value, then $\hat{x}_{t2}^m$ is said to be a false neighbour of $\hat{x}_{t1}^m$. In the FNN method one should choose the value, for which the amount of false nearest neighbours is minimised.

A criterion of *lag* determination which is often used, is a method called mutual information – MI, focusing on the investigated system as a producer of information. The aim of this method is to evaluate an amount of information about the state $s(t + lag)$, which may be forecasted according to information included in the state $s(t)$ (see Łażewski, Zator (2002)). To calculate this, Fraser and Swinney (1986) proposed a "mutual information" function and in addition they suggested that when its first minimum occurs, the proper *lag* is determined.

3. A local linear approximation

Unlike truly random processes, very precise short-term predictions of chaotic dynamics are possible. Takens theorem implies that for $m \geq 2d+1$ there exists the function $g_T : R^m \to R$, satisfying:

$$x_{t+T} = g_T(\hat{x}_t^m) \equiv g_T(x_t, x_{t-lag}, \ldots, x_{t-(m-1)lag}) . \tag{3}$$

It means that for time series with length of N one can determine a forecast, using the formula:

$$x_{N+T} = g_T(\hat{x}_N^m) \equiv g_T(x_N, x_{N-lag}, \ldots, x_{N-(m-1)lag}) , \tag{4}$$

where T is a prediction horizon. Potentially g_T may be a complicated nonlinear function, but it can be approximated by $\tilde{g}_T$ of a standard functional form. In a local linear approximation the m-dimensional linear polynomial $\tilde{g}_T$ is considered, ie:

$$\tilde{g}_T(x_1, x_2, \ldots, x_m) = \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_m x_m . \tag{5}$$

An estimation of the coefficients α_i is proceeded by establishing a value k – a number of nearest (in the sense of the fixed m-dimensional norm) neighbours of vector $\hat{x}_N^m$. Based on the found neighbours, the polynomial is fitted through the corresponding data points, by least squares. It should be pointed out that *a priori* established value k may be smaller than the amount of all available m-histories and that is why it is called “a local approximation”. Of course, the calculated forecasts depend on the values of the parameters k, m and *lag*.

4. Forecasting results

In this paper, four chaotic time series generated by the following models have been considered:

1) Logistic map:

$$x_{t+1} = 4x_t(1 - x_t) \text{, for the initial state } x_0 = 0.7. \tag{6}$$

Logistic map is a one-dimensional discrete chaotic system, generating complex dynamics widely described in literature. It has been quite often used by theorists of economics to construct chaotic models (see Sordi (1996)).

2) Henon map:

$$(x_{t+1}, y_{t+1}) = (1 - 1{,}4x_t^{\,2} + y_t;\ 0{,}3x_t) \text{, for } (x_0, y_0) = (0{,}9;\ 0{,}9). \tag{7}$$

Henon map is an example of a two-dimensional discrete chaotic system. In this paper the time series of its first coordinates has been investigated.

3) Lorenz system:

$$\begin{aligned} dx/dt &= 16(y - x) \\ dy/dt &= -xz + 45{,}92x - y \\ dz/dt &= xy - 4z \end{aligned} \tag{8}$$

Lorenz system is a chaotic system with continuous time, proposed by a meteorologist E. Lorenz. The investigated time series has been generated by the equation $x_t = x(t \cdot 0{,}01)$, for $\left(x(0), y(0), z(0)\right) = (1,1,1)$.

4) Kaldor model.

The considered model is a discrete version of a continuous model of economic growth proposed in 1940 by N. Kaldor. It consists of two difference equations:

$$\begin{aligned} Y_{t+1} - Y_t &= \alpha\ (I_t(Y_t, K_t) - S_t(Y_t)) \\ K_{t+1} - K_t &= I_t(Y_t, K_t) - \delta K_t \end{aligned}, \tag{9}$$

where Y is income, K – capital stock, I – gross investments, S – savings and δ is the constant depreciation rate. Assuming that $S_t(Y_t) = s \cdot Y_t$ and $I_t = c \cdot 2^{-1/(dY_t + \varepsilon)^2} + e \cdot Y_t + a \cdot \left(f/K_t \right)^g$, Kaldor model may generate the strange attractor. The following set of the parameters leading to chaotic motion were applied in this paper: $\alpha = 20$, s=0.21, $\delta = 0{,}05$, a=5, c=20, d=0.01, $\varepsilon = 0{,}00001$, e=0.05, f=280, g=4,5 (see Lorenz (1989)). The time series (Y_t) generated from the initial state $Y_0 = 65, K_0 = 265$ has been forecasted.

The analyzed time series with length of 1900 were divided into two parts: a part A – the first 1715 and a part B – the last 185 observations. For each value x_i from part B (i=1716, 1717, … ,1900) the one-step-ahead forecast has been calculated:

$$\tilde{x}_i = \tilde{g}_1(\hat{x}_{i-1}^m) \equiv \tilde{g}_1(x_{i-1}, x_{i-1-lag}, \ldots, x_{i-1-(m-1)lag}). \tag{10}$$

The part A was used to obtain parameters m and lag and to estimate polynomial $\tilde{g}_1$ coefficients. Values k = m+2, … , $1714 - (m-1) \cdot lag$ have been considered. To evaluate the accuracy of the predictions, for each k, the root mean squared error:

$$\sigma = \sqrt{\frac{1}{185} \cdot \sum_{i=1716}^{1900} \left(x_i - \tilde{x}_i\right)^2} \tag{11}$$

was computed. For convenience it was normalized by the standard deviation of the data from part A, forming the normalized error (see Farmer, Sidorowich (1987)):

$$\sigma' = \sigma / \sigma_x \cdot 100\%. \tag{12}$$

Two methods of determining the parameters of delay vectors have been considered:

1) "FNN–MI":

False nearest neighbours and mutual information have been applied to part A, to calculate the embedding dimension m and the time delay lag.

2) "PROG":

The part A has been divided into two parts: A_1 – the first 1650 and A_2 – the last 65 observations. For each observation from A_2 a local linear approximation has been used to forecast its value. The estimation of polynomial coefficients has been made, based on part A_1. A forecast has been made for the following set of parameters: lag=1,2,…5, m=1,2,…,15, k=m+2, … , $1649-(m-1)\cdot lag$. Then the values of parameters leading to the smallest prediction error have been chosen.

The smallest prediction errors with an adequate combination of parameters are summarized in Tables 1–4. For comparison, the errors obtained from ARMA models are also given. The illustrations of the root mean squared errors as a function of *k* are given in Figures 1–4.

Table 1. The smallest prediction errors for the logistic map

Method	σ	σ'	**Parameters**	**Optimal *k***
LA PROG	**10^{-7}**	**10^{-5}%**	*m*=2, *lag*=1	*k*=6
LA FNN–MI	**10^{-5}**	**0.002%**	*m*=1, *lag*=7	*k*=4
ARMA	**0.33**	**96.03%**	White noise	

Source: Author's calculations.

Table 2. The smallest prediction errors for the Henon map

Method	σ	σ'	**Parameters**	**Optimal *k***
LA PROG	**10^{-4}**	**0.01%**	*m*=4, *lag*=1	*k*=7
LA FNN–MI	**0.19**	**25.50%**	*m*=1, *lag*=18	*k*=16
ARMA	**0.63**	**85.59%**	ARMA(2,6)	

Source: Author's calculations.

Table 3. The smallest prediction errors for the Lorenz system

Method	σ	σ'	**Parameters**	**Optimal *k***
LA PROG	**0.004**	**0.03%**	*m*=14, *lag*=1	*k*=34
LA FNN–MI	**0.383**	**3.00%**	*m*=15, *lag*=10	*k*=107
ARMA	**0.008**	**0.06%**	ARMA(5,4)	

Source: Author's calculations.

Table 4. The smallest prediction errors for the Kaldor model

Method	σ	σ'	**Parameters**	**Optimal *k***
LA PROG	**0.11**	**0.41%**	*m*=2, *lag*=1	*k*=6
LA FNN–MI	**1.86**	**7.13%**	*m*=3, *lag*=3	*k*=13
ARMA	**17.28**	**66.45%**	ARMA(2,3)	

Source: Author's calculations.

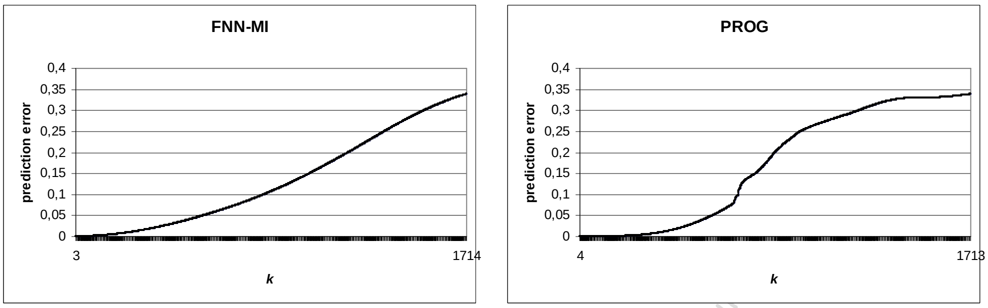

Fig. 1. Prediction errors for the logistic map
Source: Author's calculations.

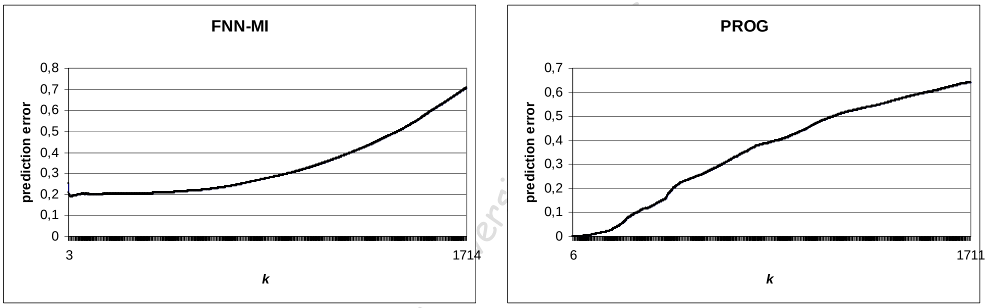

Fig. 2. Prediction errors for the Henon map
Source: Author's calculations.

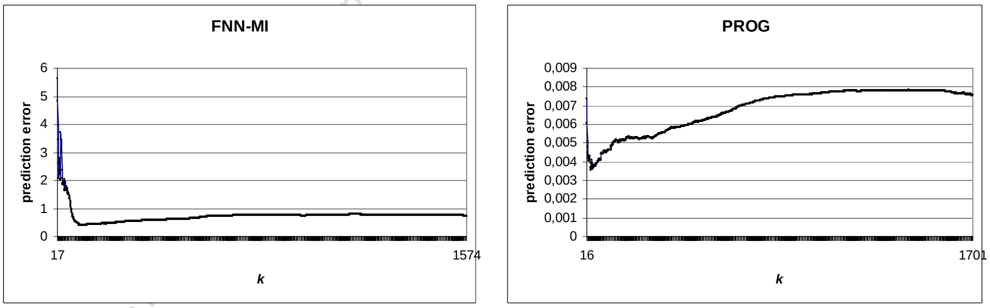

Fig. 3. Prediction errors for the Lorenz system
Source: Author's calculations.

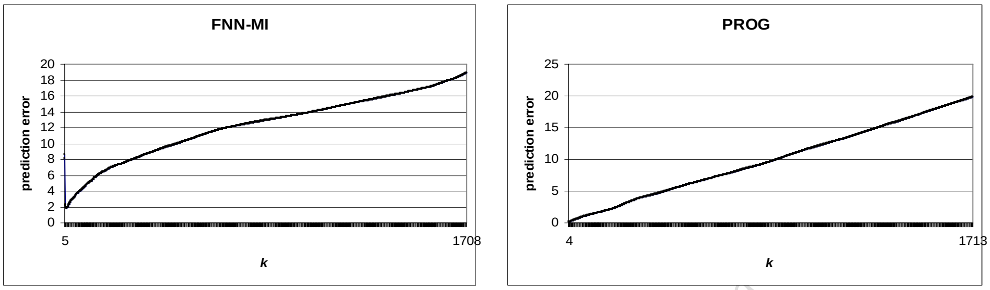

Fig. 4. Prediction errors for the Kaldor model
Source: Author's calculations.

The differences and the quotients of prediction errors for the parameters obtained from the "FNN–MI" and "PROG" procedures are illustrated in Figures 5–8. A positive difference indicates an advantage of the "PROG" procedure. The quotients charts illustrate how many times the prediction error obtained from "PROG" is smaller than the obtained one from "FNN–MI". It has been shown that the most accurate forecasts of chaotic motions are obtained from a local linear approximation for small values of k (see Castagli (1992)). That is why, only $k \leq 150$ have been marked.

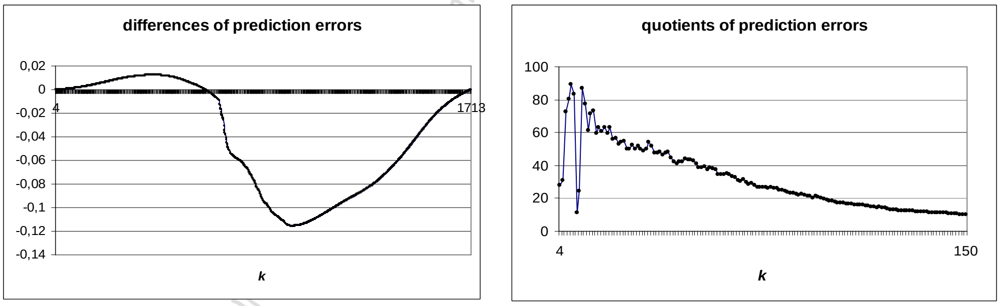

Fig. 5. The differences and the quotients of prediction errors for the logistic map
Source: Author's calculations.

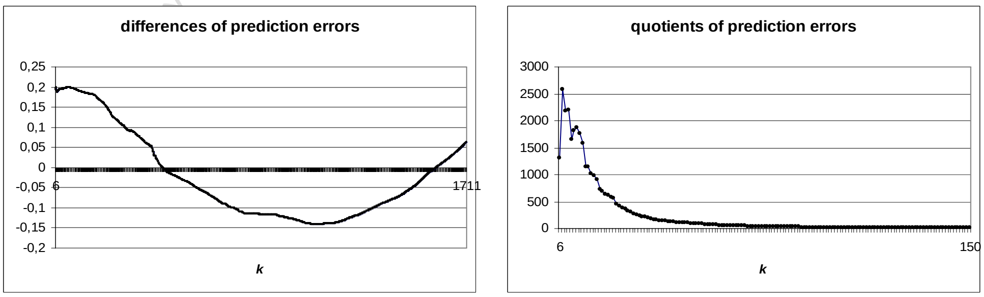

Fig. 6. The differences and the quotients of prediction errors for the Henon map
Source: Author's calculations.

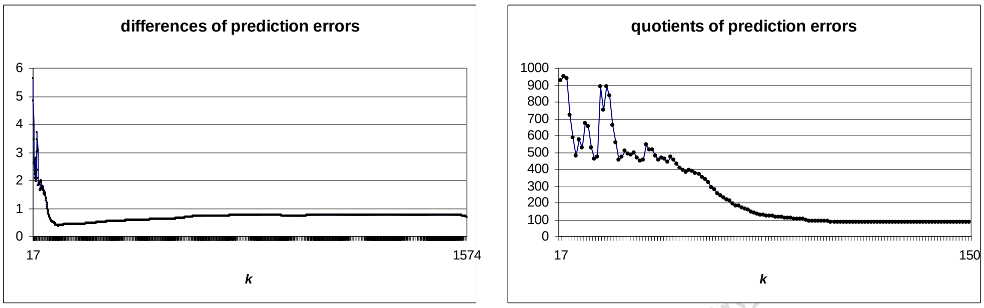

Fig. 7. The differences and the quotients of prediction errors for the Lorenz system
Source: Author's calculations.

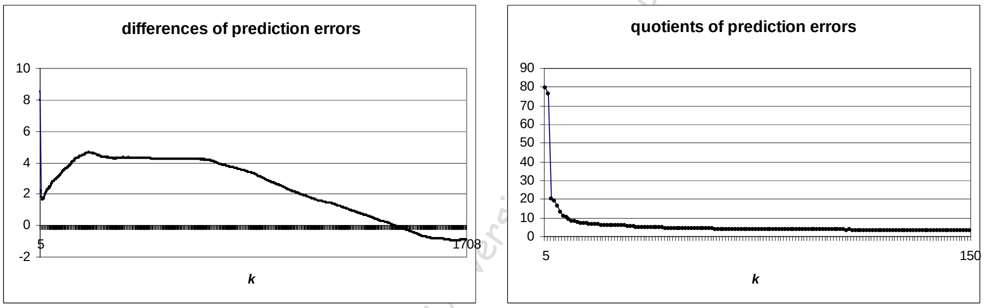

Fig. 8. The differences and the quotients of prediction errors for the Kaldor model
Source: Author's calculations.

Table 5 summarizes values of k for which procedure "PROG" is superior and the smallest and the biggest quotients for $k \leq 150$.

Table 5. A comparison of "PROG" and "FNN–MI" procedures

Time series	**An advantage of "PROG" procedure**	**A range of quotients of prediction errors**
Logistic map	k<621	from 10.09 to 89.30
Henon map	k<452	from 10.43 to 2590.65
Lorenz map	every k	from 83.21 to 953.66
Kaldor model	k<1421	from 3.36 to 79.59

Source: Author's calculations.

5. Conclusions

The results obtained in this paper prove that a very accurate short-term prediction of chaotic time series is possible. In application to analyzed time series, a local linear approximation has resulted in much better predictions than the ARMA models. The only exception was the Lorenz series, where the superiority depended on an applied procedure of determining the parameters of delay vectors.

The accuracy of forecasts obtained from a local linear approximation depended highly on the applied method of establishing the parameters of delay vectors. For each time series, the "FNN–MI" procedure has led to much worse predictions than "PROG". The results show that prediction errors also depend highly on a value of the third parameter: a number of nearest neighbours *k*, used in estimating the approximation polynomial coefficients. In the case of chaotic time series, the most accurate forecasts are obtained from a local linear approximation for small values of *k*. It has been shown, that for such values the superiority of the "PROG" procedure is clearly seen. It implies that FNN and MI are not adequate methods of determining the parameters of delay vectors for forecasting.

References

Bask, M. (1998), Esseys on Exchange Rates: Deterministic Chaos and Technical Analysis, *Working paper*, Umea University, 1–36.

Castagli, M. (1991), Eubank S., Farmer J. D., Gibson J., State Space Reconstruction in the Presence of Noise, *Physica* D, 51, 52–98.

Casdagli, M. (1992), Chaos and Deterministic versus Stochastic Non-linear Modelling, *Journal of the Royal Statistical Society* B, 54, no. 2, 303–328.

Farmer, J. D., Sidorowich, J. J. (1987), Predicting Chaotic Time Series, *Physical Review Letters*, 59, 845–848.

Fraser, A. M., Swinney, H. L. (1986), Independent Coordinates for Strange Attractors from Mutual Information, *Physical Review* A, 33, 1134–1140.

Kennel, M. B., Brown, R., Abarbanel, H. D. I. (1992), Determining Embedding Dimension for Phase-space Reconstruction Using a Geometrical Construction, *Physical Review* A, vol. 45, no. 6, 3403–3411.

Lorenz, H–W. (1989), *Nonlinear Dynamical Economics and Chaotic Motion*, Springer--Verlag, Berlin Heidelberg.

Łażewski, M., Zator, K. (2002), Analiza chaosu deterministycznego w szeregach czasowych cen akcji pozbawionych trendu a prognozowanie rynków finansowych (Analysis of Chaos in Detrended Stock Price Series and Prediction of Financial Data), in: *Rynek kapitałowy. Skuteczne inwestowanie*. (*Capital market. Efficient investing*), ed. W. Tarczyński, Szczecin, 333–344.

Sordi S. (1996), Chaos in Macrodynamics: an Excursion Through the Literature, presented at *Perspectives in Macroeconomics 12–13 04*, 1–27.

Takens, F. (1981), Detecting Strange Attractors in Turbulence, (D. Rand and L.Young, Eds), in: *Dynamical Systems and Turbulence*, Springer-Verlag, 366–381.

Zawadzki, H. (1996), Chaotyczne systemy dynamiczne (Chaotic Dynamic Systems), Wydawnictwo Akademii Ekonomicznej im. Karola Adamieckiego, Katowice.

Zeng, X., Pielke, R. A., Eykholt, R. (1992), Extracting Lyapunov Exponents From Short Time Series of Low Precision, *Modern Physics Letters* B, 6, 55–75.