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Stochastic Unit Roots Processes – Identification and Application*

1. Introduction

A standard way to model macroeconomic and finance series is a fixed-coefficient model as ARIMA model, for example, presented by Box and Jenkins (1976). Most popular way of modeling is using I(1) representation of stock prices via a random walk model. It appears however that, recent empirical test results (Granger, Swanson (1997); Sollis at al. (2000)) suggest that macroeconomic and financial time series are often processes that have a root that is not constant, but is stochastic. The stochastic unit root processes (STUR) are non-stationary and do not become stationary after taking differences of any order. It can be shown that the process that has an exact unit root, also has stochastic one.

This class of processes was considered in the articles by Leybourne, McCabe, Mills (1996), Leybourne, McCabe, Tremayne (1996) and Granger, Swanson (1997). The models describing stochastic unit root processes belong to a wide class of time-varying parameters models and their state space representation can be easily written.

The paper is organized as follows: sections 2 and 3 present the model and its properties, the fourth part includes some useful information about sample properties of ML estimator. In the fifth part the estimated STUR models for the returns of index WIG20 are shown. Conclusions close the paper in section six.

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2. The model and ML estimation

One of possible representations of the STUR process (*stochastic unit roots*), is the following

$$y_t = \alpha_t y_{t-1} + \varepsilon_t , \tag{1}$$

where:

$$\alpha_t = \alpha_0 + \delta_t ,$$
$$\delta_0 = 0 ,$$
$$\delta_t = \rho\delta_{t-1} + \eta_t , \tag{2}$$

and also $|\rho| \le 1$. Stochastic processes $\varepsilon_t \sim N(0,\sigma^2)$ and $\eta_t \sim N(0,\omega^2)$ are assumed to be independent.

For $\alpha_0 = 1$ and $\omega^2 = 0$, y_t is random walk process. For $\alpha_0 = 1$ and $\omega^2 > 0$, we have a process with a unit root in mean, called a stochastic unit root process.

Model (1)–(2) may be written as follows:

$$\Delta y_t = \delta_t y_{t-1} + \varepsilon_t , \tag{3}$$
$$\delta_t = \rho\delta_{t-1} + \eta_t , \tag{4}$$

where y_t denotes an observed process at time *t*. Here ε_t and η_t are white noise processes having zero mean and respective variances σ^2 and ω^2. In addition ε_t is independent of η_t. Equation (3) can be rewritten in an equivalent form, i.e.:

$$y_t = (1 + \delta_t) y_{t-1} + \varepsilon_t . \tag{5}$$

When $\rho = 0$ and $\omega^2 = 0$ then parameter δ_t is zero for all *t* and y_t is a standard random walk process.

The state space representation of the above model is straightforward. The Kalman filter can be obviously used for its estimation. Assuming normality, the state space model can be written as (Harvey (1989); Hamilton (1994)):

$$z_t = H_t^{'} \xi_t + w_t , \tag{6}$$
$$\xi_t = F\xi_{t-1} + v_t . \tag{7}$$

In the state space representation, equation (6) is called the observation equation, and (7) is the state equation. Thus, z_t is a $(n \times 1)$ vector of observations at time

t, ξ_t is a state vector of dimension $(r \times 1)$. Furthermore, $H_t^{'}$ is a $(n \times r)$ observation matrix, F is a $(r \times r)$ transition matrix. The disturbances, w_t and v_t are assumed to be mutually and serially uncorrelated, i.e.:

$$E\left(w_t w_\tau^{'}\right) = \begin{cases} R & dla\ t = \tau \\ 0 & dla\ t \neq \tau \end{cases} \quad \text{and} \quad E\left(v_t v_\tau^{'}\right) = \begin{cases} Q & dla\ t = \tau \\ 0 & dla\ t \neq \tau \end{cases},$$

where R and Q are $(n \times n)$ and $(r \times r)$ matrices, respectively.
Let $\hat{\xi}_{t-1}$ denote expectation of a state vector ξ_t conditional on all information available at time $t-1$, and W_{t-1} is a mean squared error matrix:

$$W_{t-1} = E\left[\left(\hat{\xi}_{t-1} - \xi_{t-1}\right)\left(\hat{\xi}_{t-1} - \xi_{t-1}\right)\right]. \tag{8}$$

The Kalman filter equations for updating from time $t-1$ to time t are:

$$\hat{\xi}_{t|t-1} = F\hat{\xi}_{t-1}, \tag{9}$$

$$W_{t|t-1} = E\left[\left(\xi_t - \hat{\xi}_{t|t-1}\right)\left(\xi_t - \hat{\xi}_{t|t-1}\right)\right] = FW_{t-1}F' + Q. \tag{10}$$

$$\hat{z}_{t|t-1} = H_t^{'}\hat{\xi}_{t|t-1} \tag{11}$$

$$u_t = z_t - \hat{z}_{t|t-1} \tag{12}$$

$$K_t = E\left(u_t u_t^{'}\right) = H_t^{'} W_{t|t-1} H_t + R. \tag{13}$$

$$\hat{\xi}_t = \hat{\xi}_{t|t-1} + W_{t|t-1} H_t K_t^{-1}\left(z_t - H_t^{'}\hat{\xi}_{t|t-1}\right), \tag{14}$$

$$W_t = W_{t|t-1} - W_{t|t-1}^{'} H_t K_t^{-1} H_t^{'} W_{t|t-1}. \tag{15}$$

For equations (3)–(4), the state space model depends upon unknown parameters $\theta = \left(\rho, \sigma^2, \omega^2\right)$. In this case, θ can be estimated by the maximum likelihood method, which is usually implemented in the filtering algorithm. The exact loglikelihood is easily derived from the Kalman filter. For observation t it is given by:

$$\ln L_t = -\frac{n}{2}\ln(2\pi) - \frac{1}{2}\ln\left|W_t\right| - \frac{1}{2}u_t^{'} W_t^{-1} u_t \tag{16}$$

Estimators of θ are obtained numerically by maximising the expression $L = \sum_{t=1}^{T} \ln L_t$.

3. Testing for the stochastic unit roots

Leybourne, McCabe and Tremayne (1996) have proposed a testing procedure (LMT hereafter), where under the alternative the stochastic unit root is assumed (see also Leybourne. McCabe and Mills (1996)). Hypotheses in the LMT test consider the variance ω^2 characteristics in equation (4). The null is $H_0 : \omega^2 = 0$, what means the random walk process or ARIMA(p,1,0), while the alternative is as follows $H_1 : \omega^2 > 0$.

To avoid the influence of the deterministic trend and the autocorrelation, the model can include the linear or quadratic time trend, and the autoregressive lags of the dependent variable, so it takes the following form:

$$y_t^* = \alpha_t y_{t-1}^* + \varepsilon_t , \tag{17}$$

where

$$y_t^* = y_t - P_t - \sum_{t=1}^{p} \varphi_i y_{t-1} , \tag{18}$$

where P_t means a deterministic component, for example the trend: $P_{1t} = \beta + \gamma t + \theta t(t+1)/2$ or $P_{2t} = \beta + \gamma t$. The autoregressive part in (18) is stationary and its role is similar to the augmentation in the Augmented Dickey Fuller test.

If in H_1 $|\rho|<1$, then Z statistics is computed in the following way:

1. estimating the equation (19) using OLS

$$\Delta y_t = \Delta P_t + \sum_{i=1}^{p} \varphi_i \Delta y_{t-i} + \varepsilon_t . \tag{19}$$

2. computing the statistics:

$$Z = T^{-\frac{3}{2}} \sigma^{-2} \kappa^{-1} \sum_{t=1}^{T} \left(\sum_{j=1}^{t-1} \varepsilon_j \right)^2 \left(\varepsilon_t^2 - \sigma^2 \right), \tag{20}$$

where:

$$\sigma^2 = T^{-1} \sum_{t=1}^{T} \varepsilon_t^2 \text{ and } \kappa^2 = T^{-1} \sum_{t=1}^{T} \left(\varepsilon_t^2 - \sigma^2 \right)^2 .$$

Choosing one of the trend model P_{1t} or P_{2t} we denote the respective statistics Z_1 and Z_2 Examining the effect of overfitting, Leybourne, McCabe and Mills (1996) and Leybourne, McCabe and Tremayne (1996), showed that presented statistics are mostly robust for fitting redundant lags in Δy_t. Moreover, when $\hat{\varepsilon}_t$ are GARCH process, it appears that conditional heteroskedasticity in the residuals does not cause a substantial power reduction. However it does not hold when residuals are IGARCH process. (Granger, Swanson (1997)). The empirical critical values of Z for various values of T are reported in Table 1.

Table 1. The critical values for the LMT statistics

T	$p = 0{,}01$	$p = 0{,}05$	$p = 0{,}1$
50	0,349	0,215	0,161
100	0,320	0,192	0,142
250	0,289	0,168	0,122
500	0,278	0,161	0,114
1000	0,261	0,149	0,104

Source: Leybourne, McCabe, Tremayne (1996) and Granger, Swanson (1997).

4. Sample properties of the ML estimator

This section presents sample properties of maximum likelihood estimator based on Monte Carlo simulation. In order to examine properties 1000 realisations of the STUR process were generated (equations (3)–(4)), then estimates of θ were obtained numerically by maximising the likelihood function. The data used for simulations contained 100, 250 and 500 observations. For every parameter, the point estimate of the variation coefficient has been computed. Variation coefficient was calculated as follows: $D(\hat{\theta})/E(\hat{\theta})$, where $D(\hat{\theta})$ is standard deviation and $E(\hat{\theta})$ is a mean of the sample estimates. The detailed outcomes are presented in Table 2. Moreover the bias of the sample estimator has been computed as $\lfloor E(\hat{\theta})/\theta \rfloor - 1$, where θ denotes a vector of true values of parameters.

Analyzing results presented in Table 2, we can claim that the maximum likelihood estimation technique gives satisfactory estimates, esspecially for large samples; $T = 500$. The estimates of variance of the disturbances in observation equation σ^2 are the most accurate.

The estimates of variance of the state equation ω^2 are the most imprecise. They are much more inaccurate than for other parameters, especially in the case of small sample; $T = 100$. Numerical value of the parameter ω^2 has essential impact on accuracy of other estimates. The comparison of computed values shows that the less numerical value of ω^2, the more biase and variance of the estimates (see Table 2).

5. STUR models for WIG20 (Polish stock index)

In the presented paper, daily and weekly returns on Polish stock index WIG20 listed on the WSE were considered to be examined whether they have the stochastic unit roots or not. The observed period contains respectively 2421 daily observations from July 1994 till March 2004 and 380 weekly observations, in the same period of the time.

Concerning daily data we identified stochastic unit root with Z statistics equal to 0.3085, while for weekly returns Z statistics was 0,219. These test values suggest that both – daily and weekly – returns of WIG20 have a stochastic unit root. The results of the estimation are given in Table 3.

Table 3. The results of estimation STUR models for WIG20.

Daily returns of WIG20	Weekly returns of Wig20
ω^2 = 3.15236E–04 σ^2 = 197.445	ω^2 = 1.53454E–06 σ^2 = 2.148

The t-statistics values for the autoregressive coefficients suggest, that they are not significantly different from zero at any conventional significance level and therefore ρ is omitted in the fitted model. The stochastic parameter $\alpha_t = \alpha_0 + \delta_t$ estimates, for $\alpha_0 = 1$ are shown in Fig. 1 and 2 respectively. The results are similar to those presented in Sollis at al. (2000) for chosen stock indices.

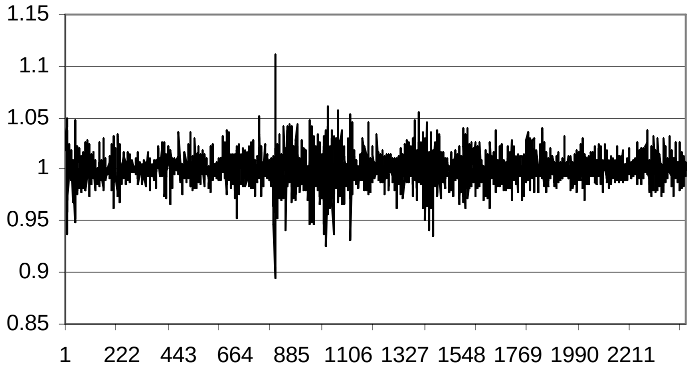

Fig. 1. The stochastic parameter $\alpha_t = 1 + \delta_t$ estimates for daily returns of Wig20

Source: Authors' computations.

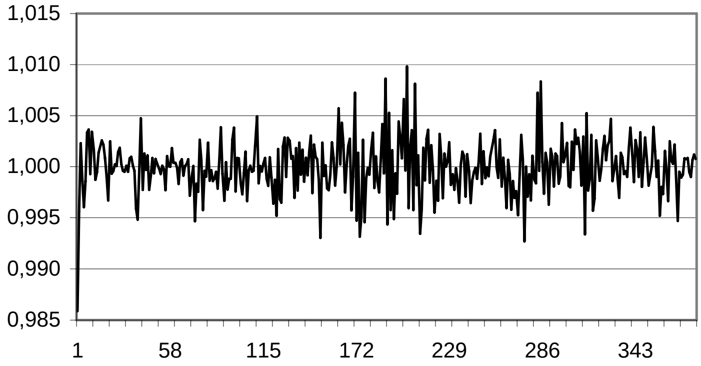

Fig. 2. The stochastic parameter $\alpha_t = 1 + \delta_t$ estimates for the weekly returns of Wig20
Source: Authors' calculations.

The most important output results are those concerning variances of state and observation equations. Analyzing the results of estimation we can state that the variability of daily returns is greater than observed for weekly returns. Despite the number of the observations it is consistent with the empirical facts. The daily returns are – in normal market conditions – usually more volatile that returns observed in longer periods. The figures show that the analyzed models cover to some extent clustering in the variance observed in daily as well as in weekly rates of return on WIG20.

6. Conclusions

We analyzed a simple form of stochastic unit roots representation. The model belongs to the time-varying parameters class of models. We found that the state space form is most convenient for its formulation. Consequently we used the Kalman filter to estimate it. We found that some financial time series – represented here by WIG20 – are better characterized by the stochastic unit root model than by the exact unit root process.

Finding of the stochastic unit roots in the economic time series extents our perception of real processes and shows the mechanism of their changes. It also gives a useful information of the limits of the standard unit roots tests.

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Table 2. ML estimates obtained by Monte Carlo simulation (1000 replications, sample length: 100, 250 i 500). Variation coefficient is calculated as follows: $D(\hat{\theta})/E(\hat{\theta})$, (standard text) and bias of sample estimator has been computed as $\left|E(\hat{\theta})/\theta\right|-1$ (italic text).

	$T=100$	$T=250$	$T=500$		$T=100$	$T=250$	$T=500$
$\rho=0{,}2$ $\omega^2=0{,}01$ $\sigma^2=1$	2.243; *-0.085* 0.835; *0.140* 0.237;*-0.049*	1.332; *-0.103* 0.377; *-0.024* 0.155; *0.004*	0.620; *-0.063* 0.200; *-0.017* 0.123; *0.028*	$\rho=0{,}2$ $\omega^2=0{,}001$ $\sigma^2=1$	2.775;*-0.056* 1.614; *4.972* 0.188; *-0.137*	2.635; *-0.112* 1.395; *0.703* 0.130; *-0.043*	1.716; *0.021* 0.816; *0.086* 0.097; *0.012*
$\rho=0{,}6$ $\omega^2=0{,}01$ $\sigma^2=1$	0.606; *-0.109* 0.791; *0.111* 0.241; *-0.077*	0.283; *-0.039* 0.320; *0.019* 0.168; *-0.000*	0.108; *-0.026* 0.174; *0.044* 0.138; *0.015*	$\rho=0{,}6$ $\omega^2=0{,}001$ $\sigma^2=1$	0.764; *-0.115* 2.035; *3.917* 0.1869; *-0.130*	0.616; *-0.089* 1.229; *0.467* 0.126; *-0.042*	0.410; *-0.045* 0.685; *0.066* 0.095; *-0.017*
$\rho=0{,}9$ $\omega^2=0{,}01$ $\sigma^2=1$	0.222; *-0.096* 0.799; *0.319* 0.335; *-0.031*	0.054; *-0.044* 0.461; *0.415* 0.289; *-0.012*	0.031; *-0.033* 0.329; *0.512* 0.213; *-0.007*	$\rho=0{,}9$ $\omega^2=0{,}001$ $\sigma^2=1$	0.321; *-0.123* 2.191; *1.698* 0.198; *-0.134*	0.164; *-0.049* 1.118; *0.250* 0.130; *-0.050*	0.063; *-0.018* 0.293; *0.055* 0.109; *-0.028*

Source: Authors' calculations.