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Kalman Filters and Specification Errors of Hyper-Structure

The article refers to the use of Kalman filters in analyses of econometric models with time-varying parameters. These models were considered earlier in Grzesiak (1995), (1997), (1999) and also in other papers by the same author. In the above-mentioned articles it was stated that a hyper-structure of the linear stochastic and dynamic system presented below:

$$y_t = x_t^T \alpha_t + \varepsilon_t, \quad t = 1,2,...,n, \tag{1}$$

$$\alpha_t = F\alpha_{t-1} + Gu_t + H\eta_t, \quad t = 1,2,...,n, \tag{2}$$

consisting of transition matrix F, reaction matrix G, innovation matrix H as well as covariance matrix Q for η_t and variance σ^2 is known[1].

However in the empirical analysis of time-varying structural parameters, the hyper-structure is not known and has to be determined from the available data. Such a situation is defined as the identification of a parameter process. It can be formally divided into three stages: first the type of parameter process has to be defined – whether it is stationary or non-stationary. Then the structure of the parameter process has to be determined. Initially we do not know, for example, what order of auto-regression fits in with the particular components of the parameters vector, which means that both the dimension and structure of transition matrix F are not known. And finally the unknown parameters of matrixes F, G, H and Q as well as variance σ^2 have to be estimated.

Significant is the fact that, in the identification of a parameter process, the order of the particular steps or stages is unimportant. We assume that to determine the type and structure of a parameter process, alternative models of parameters have to be estimated, which makes the empirical analysis very impor-

[1] Definition of matrix Q was presented in Grzesiak (1995), p. 53.

tant. Therefore, we will now assess the unknown parameters of the hyper-structure, assuming that the structure of the parameter process is known, e.g. we will consider the third step of the identification problem first.

In our opinion, econometric literature referring to this subject is hardly convincing. In empirical analyses, the authors used and recommended the “random walk” model (see: for example Athans (1974), Brännäs (1981), Haas (1983), Otter (1978) and Szeto (1973)):

$$\beta_t = \beta_{t-1} + \eta_t, \quad t = 1,2,...,n, \tag{3}$$

$$\alpha_t = \beta_t. \tag{4}$$

It means that empirical research was limited to a non-stationary parameter model with $F = H = I$ and $G = 0$. In such a situation only covariance matrix Q and variance σ^2 must be defined. Additionally, it is assumed that matrix Q is diagonal. It is a very restrictive assumption because, in the case of “random walk” models, it means that there is no correlation between variances of the particular components of the parameters vector β_t.

However, the same causes of changes of parameters exist in many economic relations. This has led to the realisation – when doing research on consumption – that changes of consumption behaviour are matched with opposing changes in attitudes towards saving. Yet it cannot be properly reflected in a consumption function described by the “random walk” parameter model with the diagonal matrix Q.

In the widely discussed approach to the determination of diagonal covariance matrix Q, the main focus is on heuristic presentation, such as *ad hoc* specifications, or on a presentation based on a sensitivity analysis[2]. The fact that a sensitivity analysis can require a relatively large expenditure of computational time is not taken into consideration. It particularly refers to the situation when errors covariance matrix $\Sigma_{0|0}$ of the initial value and variance σ^2 are determined by means of a sensitivity analysis.

Such a “sampling approach” has found wide application. Yet it can only be justified if the Kalman filter turns out to be very resistant to the specification errors of the hyper-structure. What are the results of a resistance analysis, both theoretical in character and based on simulation studies?

The most important results in the theoretical analysis of errors can be found in the work by A. Jazwinski[3], who presented how to describe the influence of specification errors $\sum_{0|0}$, Q and σ^2 on errors covariance matrixes $\sum_{t|t-1}$ and $\sum_{t|t}$.

Theorem 1.

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[2] See: Alhans (1974), Haas (1983) Otter (1978) for example.

[3] See: Jazwinski (1970), p. 244.

Let Q_c and σ_c^2 be specified for the covariance matrix Q and variance σ^2. Let $\sum_{0|0}^C$ be the specification of the error covariance matrix $\sum_{0|0}$. Let $\sum_{t|t-1}$ and $\sum_{t|t}$ be error covariance matrixes calculated on the basis of real values of Q, σ^2 and $\sum_{0|0}$. Because $Q \le Q_c$ and $\sum_{0|0} \le \sum_{0|0}^C$ occurs, therefore for errors covariance matrixes $\sum_{t|t-1}^C$ and $\sum_{t|t}^C$ calculated together with Q_c, σ_c^2 and $\sum_{0|0}^C$:

$$\textstyle\sum_{t|t-1}^C \ge \sum_{t|t-1}, \tag{5}$$

$$\textstyle\sum_{t|t}^C \ge \sum_{t|t}. \tag{6}$$

On the strength of the above-obtained results, the following practical approach is recommended: the unknown covariance matrixes Q and $\sum_{0|0}$, as well as variance σ^2 are fixed by “conservative”, i.e. sufficiently large specifications of Q_c, $\sum_{0|0}^C$ and σ_c^2. The Kalman filter provides us with suboptimal estimations of $\alpha_{t|t}^C$ and $\sum_{t|t}^C$ for a linear stochastic, dynamic system.

Depending on the size of the errors covariance matrix it can eventually be decided whether specifications of Q_c, $\sum_{0|0}^C$ and σ_c^2 provide us with “proper” estimations of $\alpha_{t|t}^C$ or whether Q_c, $\sum_{0|0}^C$ and σ_c^2 can be replaced by less “conservative” specifications. These recommendations incline us to perform a sensitivity analysis when fixing the elements of a hyper-structure.

The following question arises: Is this method of approach also suitable for econometric applications of the Kalman filters?

More explanations for the influence of hyper-structure specification errors on the results of the Kalman filter algorithm are provided by simulation studies. The results of two of such studies will be presented below. In the first case, the research was carried out by A. McWhorter[4] for the Markov parameter model where:

$$\beta_t = A\beta_{t-1} + \eta_t \tag{7}$$

and the following results were obtained:

Estimates of the Kalman filter $y_{t|t}$ of the dependent variable y_t appear to be very resistant to errors of specification of Q, σ^2, F and initial value a_0. However, when analysing the resistance of estimates of $\beta_{t|t}$, it appears to be definitely resistant to specification errors of the hyper-structure and initial value, where specification errors of transition matrix F and initial value a_0 have par-

[4] Se McWhorter, Spivey, Wrobleski (1976), p. 280.

ticularly significant consequences. It motivates McWhorter to the following recommendations:

Because during the empirical analysis of time-varying parameters there is no possibility of checking the correctness of the estimation of $\beta_{t|t}$ – the real courses of β_t are not known in any simulation study – exercising caution with the economic interpretation of the results of estimating $\beta_{t|t}$ is highly recommended. In other words, according to McWhorter, the result confirms the usefulness of the Kalman filter in the forecasting of dependent variables y_t, also, when *ad hoc* specifications of unknown models' parameters were performed.

However, there are significant doubts because only the resistance of the filtration of $y_{t|t}$ was analysed. The Kalman filter appears to be very stable because during the "actualisation" of forecasted values generally great errors of forecasts ($y_t - y_{t|t-1}$) are also well smoothed. For the estimation of the prognostic ability of the filter, only the forecasted estimated value of $y_{t|t-1}$ is useful, which was not considered in the above-mentioned research.

The fact that mere goodness of fit of the filtration of $y_{t|t}$ in the time interval *ex post* is not sufficient for the estimation of prognostic ability, is mentioned in an other work by McWhorter, Harasimham and Simonds[5]. They estimated the well-known model I from Klein (1950) again on the quarterly data from the first quarter of 1950 until the fourth quarter of 1974, both by the OLS and the 3SLS methods. Besides, time-varying parameters were estimated by means of the Kalman filters, while parameters' dynamics was described by means of the "random walk" process.

The structure of the covariance matrix Q was accepted *ad hoc*. It turned out that the Kalman filter as compared to the OLS and 3SLS methods has a little advantage at *ex ante* forecasts for one quarter. On the other hand, 4-quarterly *ex ante* forecasts obtained by both OLS and 3SLS methods exceed the forecasts of time-varying parameters in all equations of the model.

Further simulation study was performed by Brännäs and Westlund[6]. The results of these studies confirm generally the results of previous research. Errors of specifications of initial value and transition matrix F have consequences in the form of great errors at the estimation of vector of parameters β_t. As compared, the Kalman filter appears to be highly resistant to errors of the specification of covariance matrix Q and variance σ^2. However, the pattern of Brannes's and Westlund's simulation experiment should be criticised at one point. If Q and σ^2 are real specifications, then Branas and Westlund in their simulation study permanently analyse the reaction of the Kalman filter to the $Q_C = cQ$ or $\sigma_C^2 = c\sigma^2$ type specification errors, while the results of changes of the element Q ratio were not verified.

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[5] See: McWhorter, Harasimham and Simonds (1977).

[6] See: Brännäs, Westlund (1981).

The fact that the Kalman filter does not explicitly react to the proportional changes of variance and covariance, considered by the above-mentioned authors, can easily be proved by the following theorem.
Theorem 2

Let $Q_C = CQ$, $\sigma_C^2 = C\sigma^2$ and $\sum_{0|0}^C = C\sum_{0|0}$ be specified variance and covariance with $C > 0$ for the Kalman filter. Then for calculated ratios $\alpha_{t|t-1}^c$ and $\alpha_{t|t}^c$ of the state vector α_t the following occurs:

$$\alpha_{t|t-1}^c = \alpha_{t|t-1}, \tag{8}$$

$$\alpha_{t|t}^c = \alpha_{t|t}. \tag{9}$$

For covariance matrices of the errors the following occurs:

$$\sum_{t|t-1}^C = C\sum_{t|t-1}, \tag{10}$$

$$\sum_{t|t}^C = C\sum_{t|t}. \tag{11}$$

<u>Proof</u>: The theorem is proved by means of the complete induction. The beginning of the induction for $t = 1$ results directly from equations (11)–(15) of the Kalman filter[7] for $\sum_{0|0}^C = C\sum_{0|0}$ published in Grzesiak (1995). It was assumed that the thesis is true for $t - 1$. It results from the above-mentioned equation (11).

$$\alpha_{t|t-1}^c = F\alpha_{t|t-1}^c + Gu_t = F\alpha_{t-1|t-1} + Gu_t = \alpha_{t|t-1} \tag{12}$$

and from the equation of forecast of covariance (12) the following results:

$$\sum_{t|t-1}^C = F\sum_{t-1|t-1}^C F^T + HQ_C H = FC\sum_{t-1|t-1} F^T + HCQH^T = C\sum_{t-1|t-1}. \tag{13}$$

For the filter amplification vector the following results:

$$\begin{aligned} K_t^C &= \sum_{t-1|t-1}^C x_t (x_t^T \sum_{t|t-1}^C x_t + \sigma_C^2)^{-1} = \\ &C\sum_{t|t-1} x_t (x_t^T C\sum_{t|t-1} x_t + C\sigma^2)^{-1} = K_t \end{aligned} \tag{14}$$

Thesis $\alpha_{t|t}^c = \alpha_{t|t}$ results directly from the formula of the filter state (14). Also thesis (5) results from the formula of the covariance filter (15):

$$\sum_{t|t}^C = \sum_{t|t-1}^C - K_t^C x_t^T \sum_{t|t-1}^C = C\sum_{t|t-1} - K_t x_t^T C\sum_{t|t-1} = C\sum_{t|t} \tag{15}$$

Practically, not only the absolute size of covariance matrix Q, but also its structure is unknown. It is doubtful if Q is diagonal or not, and also the ratio of size of particular elements of Q is unknown.

[7] These equations are presented in Grzesiak (1995), p. 58.

A particular possibility to estimate an unknown covariance matrix Q offers the so-called Sage and Hus filter[8] often referred to in literature. It is a modification of the Kalman filter. Unknown variance and covariance are accepted as additional components of the state vector and co-estimated recursively. However it leads to a significant increase in the size of the state vector, which makes it difficult to apply in economics due to short data series. The weight of the problem is shifted into another point, because the filtration algorithm needs the initial values of σ_0^2 and Q_0 for variance and covariance, respectively.

Summing up the discussion on specification errors it should be pointed out that such widespread methods as *ad hoc* specifications and sensitivity analysis are of little use for the estimation of an unknown hyper-structure with time-varying parameters. And that is exactly what is important for the improvement of the prognostic quality of the econometric model. It is also particularly significant for the economic interpretation of an estimated course of the parameter. The course of the estimated values of the parameter $\beta_{t|t}$ is definitely dependant on the choice of the covariance matrix Q. Moreover, if the influence of specification errors of the transition matrix F is considered, we can easily imagine that without a proper criterion of estimation, practically every “demanded” course of the parameter can be estimated according to the “proper” choice of a hyper-structure.

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