DYNAMIC ECONOMETRIC MODELS
Vol. 8 – Nicolaus Copernicus University – Toruń – 2008

Mateusz Pipień
Cracow University of Economics

On the Use of the Family of Beta Distributions in Testing Tradeoff Between Risk and Return. Bayesian Analysis for WIG Excess Returns

1. Introduction

The main goal of this paper is an application of Bayesian inference in testing the relation between risk and return of the financial time series. On the basis of the Intertemporal CAPM model, proposed by Merton (1973), we built a general sampling model suitable in analysing such relationship. The most important feature of our model assumptions is that the possible skewness of conditional distribution of returns is used as an alternative source of relation between risk and return. Thus, pure statistical feature of the sampling model is equipped with economic interpretation. This general specification relates to GARCH-In-Mean model proposed by Osiewalski and Pipień (2000).

In order to make conditional distribution of financial returns skewed we considered a constructive approach based on the inverse probability integral transformation. In particular, we apply Beta distribution transformation with two free parameters; see Jones and Faddy (2003).

Based on the daily excess returns on the Warsaw Stock Exchange Index we checked the total impact of conditional skewness assumption on the relation between return and risk on the Warsaw Stock Market. Posterior inference about skewness mechanism confirmed positive and decisively significant relationship between expected return and risk.

2. An Approach to Creating Asymmetric Distributions

The unified representation of the univariate skewed distributions that we study in the paper is based on the inverse probability integral transformation;

see Ferreira and Steel (2006) for details. The family $IP=\{\varepsilon_s, \varepsilon_s:\Omega\rightarrow R\}$, with the representative density $s(.|\theta,\eta_p)$, is called the skewed version of the symmetric family I (of random variables with unimodal symmetric density $f(.|\theta)$ and distribution function F, such that the only one modal value is localised at $x=0$) if s is given by the form:

$$s(x\,|\,\theta,\eta_p)=f(x|\theta)\cdot p(F(x|\theta)\,|\,\eta_p), \text{ for } x\in R. \quad (1)$$

The asymmetric distribution $s(.|\theta,\eta_p)$ is obtained from $f(.|\theta)$ by applying the density $p(.|\eta_p)$ as a weighting function. Within the general form (1) several classes of distributions P have been imposed on some specific families of symmetric random variables; see Pipień (2006) for a review. The most important feature of our approach is, that uniform density, namely $p(y|\eta_p)=1$ for $y\in R$, retrieves symmetry in (1); i.e. $s=f$ if $p(.|\eta_p)=I_{[0,1]}$. In the empirical part of the paper, when defining skewing density p, we make use of the family of Beta distributions. The random variable $\varepsilon_p\in(0,1)$ has a Beta distribution with parameters $a>0$ and $b>0$ if the density, denoted by $Be(.|a,b)$, is given by the formula:

$$Be(y|a,b)=y^{a-1}(1-y)^{b-1}/B(a,b), \text{ for } y\in R,$$

where normalising constant $B(a,b)$ denotes the value of the Euler beta function.

3. Introducing Skewness into GARCH-in-Mean Model

Let denote by x_j the value of a stock or a market index at day j. The excess return on x_j, denoted by y_j, is defined as the difference between the logarithmic daily return on x_j in percentage points ($r_j=100\ln(x_j/x_{j-1})$) and the risk free short term interest rate (denoted by r_j^f), namely $y_j=r_j-r_j^f$. Following Engle, Lilien and Robins (1987), and Osiewalski and Pipień (2000) we consider for y_j a simple GARCH-In-Mean process, defined as follows

$$y_j=[\alpha+E(z_j)]\, h_j^{0.5}+u_j,\ j=1,2,..., \quad (2)$$

where $u_j=[z_j-E(z_j)]h_j^{0.5}$, and z_j are independently and identically distributed random variables with $E(z_j)<+\infty$. The scaling factor h_j is given by the GARCH(1,1) equation; see Bollerslev (1986):

$$h_j=\alpha_0+\alpha_1 u_{j-1}^2+\beta_1 h_{j-1}.$$

The specific form of the conditional distribution of y_j strictly depends on the type of the distribution of z_j. Initially, in model denoted by M_0, we assumed for z_j the Student-t density with unknown degrees of freedom $\nu>1$, zero mode and unit inverse precision. We denote the value of this density as $f_t(z|0,1,\nu)$. Given model M_0, $E(z_j)=0$, $u_j=z_jh_j^{0.5}$, and hence (2) reduces to the simpler form $y_j=\alpha\, h_j^{0.5}+u_j$. Let denote by $\theta=(\alpha,\alpha_0,\alpha_1,\beta_1,\nu)$ the vector of all parameters in model M_0. The following density represents conditional distribution of the excess return at time j:

$$p(y_j\,|\,\psi_{j-1},\theta, M_0)= h_j^{-0.5}\cdot f_t(h_j^{-0.5}\cdot(y_j-\alpha h_j^{0.5})|0,1,\nu),\ j=1,2,... .$$

Given model M_0 the expected excess return (conditional to the whole past $\psi_{j\text{-}1}$) is proportional to the square root of the inverse precision h_j:

$$E(y_j \mid \psi_{j\text{-}1}, \theta, M_0) = \alpha\, h_j^{0.5}. \tag{3}$$

The parameter $\alpha \in R$ captures the dependence between expected excess return and the level of risk both measured by $E(y_j \mid \psi_{j\text{-}1}, M_0, \theta)$ and the scale parameter $h_j^{0.5}$ respectively.

Now, defining model M_1, we introduce skewness into our GARCH-In-Mean model. The resulting asymmetric distribution is obtained by skewing the density of the random variable z_j according to method presented in the previous section. Asymmetric density of z_j takes the form related to the formula (1):

$$p(z \mid M_1) = f_t(z|0,1,\nu) \cdot p[F_t(z) \mid \eta_1, M_1], \text{ for } z \in R, \tag{4}$$

where $p(.\mid \eta_1, M_1)$ defines the skewing mechanism parameterised by the vector η_1, and $F_t(.)$ is the distribution function of the Student-t random variable with $\nu>1$ degrees of freedom parameter, zero mode and unit inverse precision. For the skewing density p in M_1 we assumed the Beta distribution density with two free parameters $a>0$ and $b>0$:

$$p(y \mid \eta_1, M_1) = Be(y|a,b),\ \eta_1 = (a,b),\ a>0,\ b>0.$$

In model M_1 the conditional distribution of y_j is heteroscedastic, where time varying dispersion measure h_j, defined by GARCH(1,1) specification, is a function of the whole past of the process. The degrees of freedom parameter $\nu>1$ enable for fat tails of $p(y_j \mid \psi_{j\text{-}1}, \theta, \eta_1, M_1)$. It is also possible to test whether the dataset supports conditional distribution with Gaussian-type tails (for $\nu \to \infty$). Asymmetry of the conditional distribution can be captured by the presence of a particular skewing mechanism. If $a>b$ ($a<b$), then the conditional distribution is skewed to the left (right), while in case $a=b=1$ the skewed density in (4) reduces to the simple symmetric case. Consequently restriction $a=b=1$ nests M_0 in M_1. Additionally, skewness of the distribution of z_j in M_1 generates nonzero expectation $E(z_j)<+\infty$, hence:

$$E(y_j \mid \psi_{j\text{-}1}, \theta, \eta_1, M_1) = [\alpha + E(z_j)] h_j^{0.5}, \text{ for } E(z_j) \neq 0,$$

and conditional skewness of excess returns y_j in M_1 can be interpreted as an additional source of the relationship between risk and return. Our idea fully corresponds to Harvey and Siddique (2000), who emphasize, that systematic skewness is economically important and governs risk premium.

4. Empirical Analysis

In this part we present an empirical example of Bayesian comparison of all competing specifications. We also discuss the results of the total impact of the conditional skewness assumption on the relationship between risk and return on the Warsaw Stock Exchange (WSE). Our dataset was constructed on the basis of T=2144 observations of daily growth rates, r_j, of the index of the WSE (WIG index) from January 06, 1998 till July 31, 2006. The risk free interest rate, r_j^f, used in excess return y_j, was approximated by the WIBOR overnight interest rate (WIBOR O/N instrument). Our empirical results remained practically unchanged for r_j^f calculated on the basis of the middle and long term WIBOR interest rate and also in the case r_j^f=0 for each j.

Table 1 presents decimal logarithms of the marginal data density value, as well as the posterior probabilities of M_0 and M_1. The substantial data support is attached to model M_1, making the case of conditional symmetry f y_j rather improbable in the view of the data. Comparing posterior properties of $\alpha+E(z_j)$ in both models, we checked the empirical importance of conditional skewness (imposed in M_1) on the relationship between risk and return. Initially, given M_0, $E(z_j)$=0, and the whole information about relative risk aversion coefficient is reflected in parameter α; see (3). Just like many other researchers we obtained positive, but rather weak relationship between expected excess return and the level of risk. The posterior probability $P(\alpha>0|M_0,y)$ equal 0.92 leaves considerable level of uncertainty about the true strength and the sign of tested relation. However, in case of model M_1, the posterior probability of the positive sign of $\alpha+E(z_j)$ is greater than 0.99, leaving no doubt about the sign of the relationship. Hence, we restored the positive sign of the relationship, by imposing a particular skewing mechanism on the conditional distribution of excess return. Beta distribution transformation, applied in M_1, seems to be a proper model component, making conditional distribution of y_j more sensitive to information about risk premium contained in the dataset.

Table 1. Decimal logarithms of the marginal data density values, posterior probabilities of competing specifications, and posterior analysis of the impact of the conditional skewness assumption on the relation between risk and return

	M_1 skewness imposed by Beta density with 2 parameters	M_0 conditional symmetry and hence: $E(z_j)$=0
$\log p(y\|M_i)$	-1558.41	-1559.06
$P(M_i\|y)$, i=0,1	0.8171	0.1829
$\alpha+E(z_j)$	0.2148 *0.0852*	0.0483 *0.0337*
$P(\alpha+E(z_j)>0\|M_i,y)$	0.9972	0.9201

Figure 1 presents the plot of the posterior mean of the skewing mechanism in model M_1. The qualitative analysis of the shape of the density $p(.|\eta_1,M_1)$ enables to identify the sources of skewness of the conditional distribution of y_j supported by the dataset. As seen from the figure, the considerable amount of skewness is located in the tails of the density $p(y_j\,|\psi_{j-1},\theta,\eta_1,M_1)$. It is clear, that skewness is forced by subtle distinction between left and right tail of the conditional distribution of excess return y_j. The skewing mechanism $p(.|\eta_1)$ makes conditional distribution of excess returns more leptokurtic, as the function $p(.|\eta_1)$ has its extremes on the bounds of the interval (0,1). Since the global extreme value of $p(y|\eta_1)$ is reached at y=0, the skewing mechanism forces left conditional asymmetry in M_1.

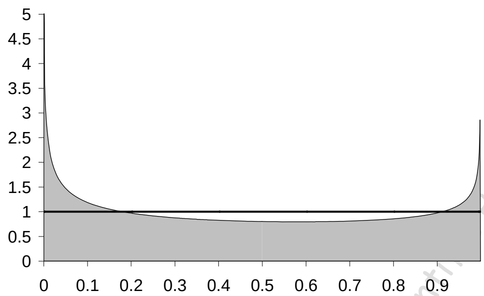

Figure 1. The plot of the posterior mean of skewing mechanism $p(.|\eta_1)$ (grey plot), together with the plot depicting the case of symmetry (black plot of the uniform density over the unit interval)

References

Bollerslev, T. (1986) Generalised Autoregressive Conditional Heteroscedasticity, *Journal of Econometrics*, 31, 307–327.

Engle, R. F., Lilien, D. M., Robins, R. P. (1987) Estimating Time-varying Risk Premia in the Term Structure: The ARCH-M Model, *Econometrica*, 55, 391–408.

Ferreira, H., Stell, M. F. J. (2006), A Constructive Representation of Univariate Skewed Distributions, *Journal of the American Statistical Association*, 101, 823–839.

Harvey, C. R., Siddique, A. (2000) Conditional Skewness in Asset Pricing Models, *Journal of Finance*, 55, 1263–1295.

Jones, M. C., Faddy, M. J. (2003) A Skew Extension of the t-Distribution, with Applications, *Journal of Royal Statistical Association B*, 65, 159–174.

Merton, R. C. (1973) An Intertemporal Capital Asset Pricing Model, *Econometrica*, 41, 867–887.

Osiewalski, J., Pipień, M., (2000) GARCH-In-Mean through Skewed t Conditional Distributions: Bayesian Inference for Exchange Rates, w: Welfe, W., Wdowiński, P. (red.), *26-th International Conference MACROMODELS'99*, Łódź , 354–369.

Pipień, M. (2006), Bayesian Comparison of GARCH Processes with Skewness Mechanism in Conditional Ditributions, *Acta Physica Polonica B*, 37, 3105–3121.