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Magdalena Osińska, Marcin Fałdziński
Nicolaus Copernicus University in Toruń

GARCH and SV Models with Application of Extreme Value Theory

1. Introduction

In the scientific research as well as in everyday life we can see a tendency to averaging many of the observed values. However, we could point out many cases where centrality measures are improper. In the case of events with extreme size the Extreme Value Theory (EVT) is appropriate. The methods of estimation in the EVT can be divided into two groups: nonparametric and parametric ones. The subject of further analysis in this paper is the *Peaks over Threshold* (POT) method, which belongs to the parametric group of the methods. The main aim of this paper is to present the application of the Extreme Value Theory in a risk analysis. We put forward a thesis that GARCH and SV models with application of the EVT can provide better estimation of the risk measures for financial time series, then standard volatility models.

2. The *Peaks over Threshold* Method in the Extreme Value Theory

The *Peaks over Threshold* method assumes that a given sequence of i.i.d. observations $X_1,\ldots,X_n$ comes from unknown distribution function F, where we are interested in excesses over a high threshold value u. Conditional excess distribution function (cedf) F_u is defined as $F_u(y)=P\left(X-u\le y\mid X>u\right)$, $0\le y\le x_F-u$, where X is a random variable, u is a given threshold, and $y=x-u$ is the excess (McNeil, Frey, 2000). The distribution F_u can be written as:

$$F_u(y)=\frac{F(u+y)-F(u)}{1-F(u)}=\frac{F(x)-F(u)}{1-F(u)} \tag{1}$$

The realizations of the random variable X lie between 0 and u, therefore the estimation of F in this interval generally poses no problems. The distribution form F_u could be found in accordance with the following theorem.

Theorem 1 (Pickands, 1975; Balkema, de Haan, 1974).

For a large class of underlying distributions F, the conditional excess distribution function $F_u(y)$, for u large, is well approximated by $F_u(y) \approx G_{\gamma,\sigma}(y)$, $u \to \infty$, where

$$G_{\gamma,\sigma}(y) = \begin{cases} 1-\left(1+\dfrac{\gamma}{\sigma}y\right)^{-1/\gamma} & \text{if } \gamma \neq 0 \\ 1-e^{-y/\sigma} & \text{if } \gamma = 0 \end{cases} \tag{2}$$

for $y \in \left[0,\left(x_F - u\right)\right]$ if $\gamma \geq 0$ and $y \in \left[0, -\sigma/\gamma\right]$ if $\gamma < 0$, where $G_{\gamma,\sigma}$ is generalized Pareto distribution. The POT method then works in the following steps:

- Select a high threshold u for a given sample $X_1,\ldots,X_n$.
- Chose N_u i.e. the number of exceeding observations $(X_{i_1},\ldots,X_{i_{N_u}})$, denoted as $Y_j = X_{i_j} - u \geq 0$.
- Fit $G_{\gamma,\sigma}$ to the excesses $Y_1,\ldots,Y_{N_u}$ to obtain estimates of γ and σ. (Beirlant, Mattys (2001)).

According to the Pickands-Balkema-de Haan theorem, for $x \geq u$, we can use the tail estimate $\hat{F}(x) = \left(1-F_n(u)\right)G_{\gamma,\mu,\sigma}(x) + F_n(u)$ to approximate the distribution function $F(x)$. It can be shown that $\hat{F}(x)$ is also generalized Pareto distribution, with the same shape parameter γ, but with scale and location parameters, correspondingly equal: $\tilde{\sigma} = \sigma\left(1-F_n(u)\right)^{\gamma}$ and $\tilde{\mu} = \mu - \tilde{\sigma}\left(\left(1-F_n(u)\right)^{-\gamma} - 1\right)/\gamma$. Thus, the POT estimator of x_p is obtained by inverting the formula for $\hat{F}(x)$. Then substituting unknown parameters of the GPD by estimates $(\hat{\gamma},\hat{\sigma})$, we get:

$$\hat{x}_p = \hat{F}^{\leftarrow}(p) = G^{-1}_{\hat{\gamma},u,\hat{\sigma}}\left(\frac{p-F_n(u)}{1-F_n(u)}\right) = u + \frac{\hat{\sigma}}{\hat{\gamma}}\left(\left(\frac{1-p}{1-F_n(u)}\right)^{-\hat{\gamma}} - 1\right) \tag{3}$$

If N_u is the number of exceedances of the threshold u and n is the total number of realizations that we have from the distribution F, the quantile estimator is:

$$\hat{x}_p = u + \frac{\hat{\sigma}}{\hat{\gamma}}\left(\left(\frac{n}{N_u}(1-p)\right)^{-\hat{\gamma}} - 1\right) \tag{4}$$

where p is close to 1. An important problem is the choice of the threshold u, because it reflects the values of estimated parameters. It is suggested in the theory, that the choice should be based on the compromise between bias and variance. In the case of higher level of threshold, we should expect to get less bias. On the other hand we get less excesses that result a higher variance.

3. *Value-at-Risk* and *Expected Shortfall* in the Extreme Value Theory

A Value-at-Risk in the EVT for the *Peaks over Threshold* method is equal:

$$VaR(\alpha) = u + \frac{\hat{\sigma}}{\hat{\gamma}}\left(\left(\frac{n}{N_u}\alpha\right)^{-\hat{\gamma}} - 1\right), \tag{5}$$

where α is a tolerance level. However the value-at-risk often fails, that is why an alternative risk measure called the *Expected Shortfall* (ES) was developed. To construct the expected shortfall, Artzner et al. (1998) defined a risk measure by means of four axioms. A risk measure that is monotonic, subadditive, positively homogeneous and translation invariant is called a coherent risk measure. From a coherency point of view, the VaR generally is not a risk measure, because it does not hold subadditivity axiom. From the theorem 1, we know, that the conditional excess distribution function for a threshold u is equal $F_u(y) \approx G_{\gamma,\sigma}(y)$. If the threshold is equal $VaR(\alpha)$, so cedf with this threshold is also the generalized Pareto distribution (GPD) with the same shape parameter, but different scale parameter. In the consequence of the equation $F_u(y) \approx G_{\gamma,\sigma}(y)$, we obtain the following formula:

$$F_{VaR(\alpha)}(y) \approx G_{\gamma,\sigma+\gamma(VaR(\alpha)-u)}(y) \tag{6}$$

By noting, that for $\gamma < 1$ the mean of the distribution is equal $(\sigma + \gamma(VaR(\alpha) - u))/(1-\gamma)$, we can retrieve the *Expected Shortfall* for the *Peaks over Threshold* method:

$$ES(\alpha) = \frac{VaR(\alpha)}{1-\gamma} + \frac{\sigma - \gamma u}{1-\gamma} \tag{7}$$

The *Expected Shortfall* is defined as $ES(\alpha) = E\left[X \mid X > VaR(\alpha)\right]$, i.e. the average loss under condition, that VaR will be exceeded. Dowd (2002) presented a way to estimate the ES, as the weighted average of the tail VaRs. For backtesting we used three tests: the failure test LR_{uc}, the mixed Kupiec-test

LR_{ind} (Haas, 2001) and the test of independence LR_{ind} (Christoffersen, 1998). However these methods do not give the opportunity to create a ranking of the models, that is why Lopez (1999) proposed to use the loss function criterion. Developing the idea, Angelidis and Degiannakis (2006) pointed out two drawbacks of Lopez's approach. Firstly, a model that does not generate any violation is recognised as the best. Secondly, a return y_{t+1} should be compared with the ES, because the VaR does not give any indication about the size of the expected loss. Therefore they proposed the following loss function:

$$\Psi_{1,t+1} = \begin{cases} \left| y_{t+1} - ES_{t+1|t} \right| & \text{if violation occurs} \\ 0 & \text{else} \end{cases} \tag{8}$$

$$\Psi_{2,t+1} = \begin{cases} \left(y_{t+1} - ES_{t+1|t} \right)^2 & \text{if violation occurs} \\ 0 & \text{else} \end{cases} \tag{9}$$

To judge which model is the best, we compute mean absolute error (MAE) $MAE = \sum_{t=1}^{\tilde{T}} \Psi_{1,t} / \tilde{T}$, and mean squared error (MSE) $MSE = \sum_{t=1}^{\tilde{T}} \Psi_{2,t} / \tilde{T}$, where $\tilde{T}$ is the number of the forecasts. Then the loss function (LF) for each model is constructed as the sum of these errors (Angelidis, Degiannakis, 2006).

4. The POT Method with Application of the Volatility Models

The existing approaches for estimating the profit/loss distribution of a portfolio of financial instruments can be schematically divided into three groups: non-parametric historical simulation methods, parametric methods based on volatility models and methods based on the Extreme Value Theory. McNeil and Frey (2000) joint all these three methods to remove their drawbacks and get out their best features. We assume that X_t is a time series representing daily observations of log return on a financial asset price, which are given by $X_t = \mu_t + \sigma_t Z_t$, where Z_t is a white noise process with zero mean, unit variance and the marginal distribution function $F_Z(z)$. We assume that μ_t is the expected return and σ_t is the volatility of the return. To implement an estimation procedure for the process X_t, we need to choose a dynamic conditional mean as well as a conditional variance model. McNeil and Frey defined simple risk measures forms for one day horizon with relation to process X_t as:

$$VaR_q^t = \mu_{t+1} + \sigma_{t+1} VaR(Z)_q \tag{10}$$

$$ES_q^t = \mu_{t+1} + \sigma_{t+1} ES(Z)_q , \tag{11}$$

where $VaR_q^t(Z)$ is the q-th quantile of Z_t and $ES_q^t(Z)$ is the corresponding expected shortfall. This approach fits a volatility model to returns. Firstly, we estimate μ_{t+1} and σ_{t+1}, and calculate model's standardized residuals. Secondly, we apply the Extreme Value Theory to estimate the $VaR_q^t(Z)$ and $ES_q^t(Z)$ with the use of the *POT* method based on the standardized residuals. In this method we stand in front of the problem to choose the number of extremes k, that we take to the estimation. Many authors have suggested several solutions but none of them has been universally adopted.

5. Simulations Results

The subject of simulations is the comparison of the estimated value-at-risk and expected shortfall measures for chosen volatility models with the models enriched by application of the EVT. The essential aspect of the comparison was to choose the best model based on the loss function proposed by Angelidis and Degiannakis (2006). In the simulations the SV model with Gaussian distribution and the GARCH model with Gaussian and t-Student error distributions were used. We have chosen the SV and GARCH models because they represent the most standard volatility models. The parameters were estimated with the maximum likelihood method in the case of the GARCH models and the *quasi*-maximum likelihood method in the case of the SV models. In the tables 1 and 2 the results of simulation carried out on daily data are shown. The time series used in simulation comprise 3000 observations (daily data: 07.11.1994 – 31.10.2006). We used 10 financial time series that represent the stock exchange returns and the foreign exchange rates. The results were similar for all time series, so we decided to show just part of them to save space[1]. For each time series a thousand VaRs and expected shortfalls were estimated, where number of extremes $k \in [20;350]$. To compute the ES for the volatility models we used Dowd's approach (Dowd (2002)), i.e. where 1000 VaRs were estimated to compute one expected shortfall. As the result the smallest values of the loss function occurred for models where the EVT was applied. Furthermore the SV-POT model is usually better for the stock exchange returns and the GARCH-POT model for the foreign exchange rates. We should draw our attention to the fact that the number of violations in the case of the EVT models is usually less then the expected one, whereas for the standard volatility models the number of violations is greater. In the case of the volatility models with the EVT it could be found this interval of k (number of extremes) values where the LR_{uc} statistics is considerably lower, but on the other hand the loss function is slightly higher. We should decide between the model that has better properties in the light of backtesting statistics and the model that better estimates the VaR and

[1] Other findings are available upon request.

ES (lover loss function). We have decided to show the models characterized by the smallest values of the loss function. In the chart 1 and 2 the VaRs and the Expected Shortfalls defined according to the SV and SV-POT models with significance level $\alpha = 0.05$ for DAX index are presented. The SV-POT models considerably better estimate log-returns in the cases, where market disturbances are significant. Second important property that could be seen in the case of the SV-POT models (generally models with the EVT) is that the VaR estimations are close to the ES values, which cannot be said about those models where the EVT was not applied. We achieved similar results for other stock returns (also from the Polish stock market), which were presented in the Polish edition of this paper (Osińska, Fałdziński, 2007). From the empirical results, it appears that the volatility models with the Extreme Value Theory better estimate the expected values of future returns, which can be very useful in the case of shock events.

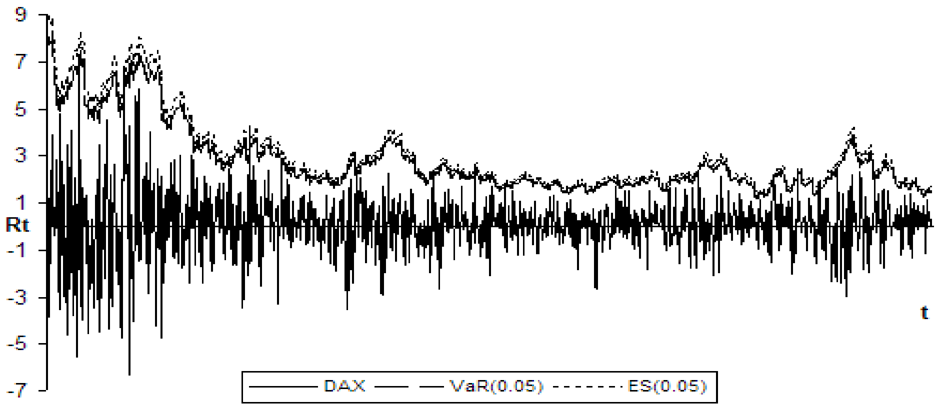

Figure 1. VaR and ES estimate from SV-POT model at confidence level $\alpha = 0.05$ for DAX index

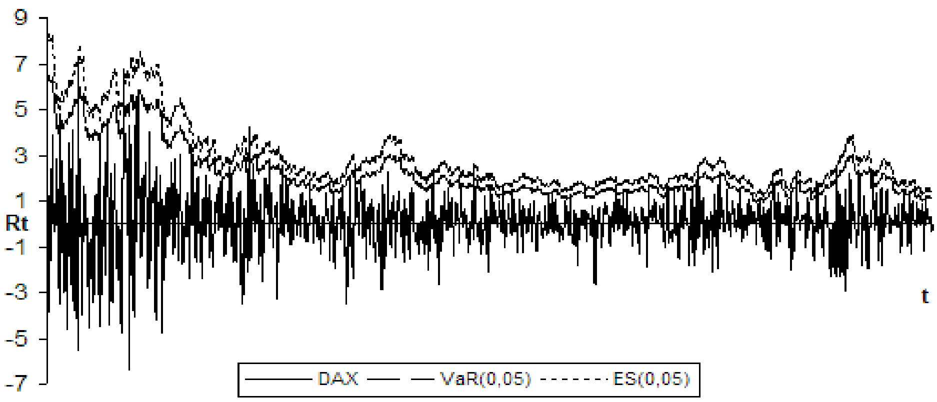

Figure 2. VaR and ES estimate from SV model at confidence level $\alpha = 0.05$ for DAX index

Table 1. Simulation results for DAX and USD-JPY index (daily data: Nov. 7, 1994 to Oct. 31, 2006, number of observations = 3000, number of iterations 1000)

DAX, $\alpha = 0.05$							
Model	k	N	LR_{uc}	$LR_{ind}K$	$LR_{ind}CH$	LF	Ranking
GARCH	-	52	0.08	80.53	0.61	0.09632	5
GARCH TS	-	53	0.19	81.55	0.50	0.13176	6
SV	-	42	1.42	45.33	0.79	0.02445	2
GARCH-POT	43	66	4.92	103.93	0.64	0.08164	4
GARCH-POT TS	38	67	5.52	104.78	0.53	0.08105	3
SV-POT	25	10	49.47	56.40	2.97	0.00467	1
DAX, $\alpha = 0.01$							
Model	k	N	LR_{uc}	$LR_{ind}K$	$LR_{ind}CH$	LF	Ranking
GARCH	-	12	0.38	27.48	0.29	0.02371	4
GARCH TS	-	10	0.00	21.91	0.20	0.01585	3
SV	-	17	4.09	28.82	4.68	0.00687	2
GARCH-POT	43	17	4.09	41.18	0.59	0.02831	6
GARCH-POT TS	38	14	1.44	1.44	0.03	0.02527	5
SV-POT	245	4	4.71	4.44	0.03	0.00057	1
USD-JPY, α=0.05							
Model	k	N	LR_{uc}	$LR_{ind}K$	$LR_{ind}CH$	LF	Ranking
GARCH	-	17	22.34	24.50	0.80	0.0179	4
GARCH TS	-	21	18.04	26.62	1.08	0.0392	6
SV	-	64	11.63	96.92	0.04	0.0203	5
GARCH-POT	65	12	36.70	32.98	0.30	0.0027	2
GARCH-POT TS	45	16	38.70	35.98	0.35	0.0026	1
SV-POT	110	24	7.04	27.18	2.38	0.0103	3
USD-JPY, $\alpha = 0.01$							
Model	k	N	LR_{uc}	$LR_{ind}K$	$LR_{ind}CH$	LF	Ranking
GARCH	-	1	13.48	11.11	0.00	0.00080	4
GARCH TS	-	1	13.48	11.11	0.00	0.00055	3
SV	-	43	76.06	98.45	3.64	0.01465	6
GARCH-POT	34	1	13.48	12.11	0.00	0.00022	1
GARCH-POT TS	120	1	13.48	13.57	0.00	0.00045	2
SV-POT	112	18	10.84	32.08	0.99	0.00588	5

Notation: k – number of extremes taken to estimation for models with EVT, N – number of violations when return is higher then VaR, LR_{uc} – failure test statistic, $LR_{ind}K$ – statistic for Kupiec's test of independence, $LR_{ind}CH$ – statistic for Christoffersen's test of independence, LF – loss function and α – significance level.

Table 2. Simulation results (daily data: Nov. 7, 1994 to Oct. 31, 2006, number of observations = 3000, number of iteration 1000)

Model ranking according to loss function for α=0.05						
Time series	GARCH	GARCH TS	SV	GARCH-POT	GARCH-POT TS	SV-POT
DAX	5	6	2	3	4	**1**
SP500	5	6	2	4	3	**1**
DJIA	4	6	5	2	3	**1**
FTSE	4	5	6	3	2	**1**
USD-JPY	4	6	5	2	**1**	3
Model ranking according to loss function for α = 0.01						
Time series	GARCH	GARCH TS	SV	GARCH-POT	GARCH-POT TS	SV-POT
DAX	4	3	2	6	5	**1**
SP500	2	6	5	4	3	**1**
DJIA	2	3	6	4	5	**1**
FTSE	3	2	6	5	4	**1**
USD-JPY	4	3	6	**1**	2	5

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