DYNAMIC ECONOMETRIC MODELS
Vol. 8 – Nicolaus Copernicus University – Toruń – 2008

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Modeling Conditional Dependencies Between Polish Financial Returns with Markov-Switching Copula Models*

1. Introduction

The study of conditional dependence structures in equity markets has recently received considerable attention among both researchers and practitioners. It is of great importance, for instance, for controlling risks, optimal portfolio choice, or analyzing volatility transmission mechanism. In this context, the issue of particular interest is asymmetry in dependence structures. One of the most spectacular examples of asymmetric behavior in equity markets is a well documented phenomenon consisting in tendency of much stronger dependence between stock returns in crisis times than those of optimistic market times. Traditional multidimensional volatility models with elliptically distributed standardized innovations cannot reproduce such dependence structures. Moreover, outside the world of elliptical distributions, for example when dealing with conditional marginals belonging to different distribution families, the standard inference about dependence using the linear correlations may be quite misleading (Embrechts *et al*., 2002). Much of the limitations encountered with traditional approach to modeling conditional dependencies between equity returns can be overcome by using the concept of copula. This tool makes possible to construct multivariate distributions with arbitrary marginals, isolating the description of the dependence structure. Copulas can also be used to investigate tail behavior in joint conditional distributions and various kinds of asymmetry. In the conditional dependence modeling a dynamics of time-varying copula has to be specified. It can be done by means of the techniques introduced by Patton (2004,

* This work was financed from the Polish science budget resources in the years 2007-2010 as the research project NN 111 1256 33.

2006) that involve some process governing the evolution of the copula parameter over time. Among alternative approaches, a combination of copula theory with regime switching models is applied (e.g. Rodriguez, 2007; Tsafack, 2006) which enables to model the conditional dependence structure with sufficient flexibility.

In this paper we model the conditional dependence structure for daily returns on the WIG20 index representing largest companies listed on the Warsaw Stock Exchange, and the MIDWIG index calculated based on shares values of medium-sized companies from the exchange. Our approach uses a Markov-switching copula model that allows us to investigate asymmetry in conditional dependencies between extremal returns on the indices. The result concerning the existence of significant asymmetry in tail dependence that we derive can be of importance for risk management.

2. Copulas and Dependence Measures

Copulas were initially introduced by Sklar (1959). Formally, an n-dimensional copula is a distribution function C on n-cube $[0,1]^n$ with standard uniform marginal distributions. Let X be an n-dimensional random variable with joint distribution function F and 1-dimensional marginal distribution functions F_i. The importance of copulas in studying of multivariate distribution functions is summarized by Sklar's theorem which states that F can be written as

$$F(x_1,\ldots,x_n) = C(F_1(x_1),\ldots,F_n(x_n)) \tag{1}$$

for some copula C. If the marginal distribution functions are continuous then C is unique and is called the copula of F or X. Conversely, if C is a copula and $F_1,\ldots,F_n$ are univariate distribution functions, then the function F defined in (1) is a joint distribution function with margins $F_1,\ldots,F_n$. An explicit representation of C in terms of F and its margins is given by

$$C(u_1,\ldots,u_n) = F(F_1^{-1}(u_1),\ldots,F_n^{-1}(u_n)), \tag{2}$$

where $F_i^{-1}(u_i) = \inf\{x_i : F_i(x_i) \geq u_i\}$. Since the marginals and the dependence structure in (1) can be separated, it makes sense to interpret the copula C as the dependence structure of F.

If a copula C is absolutely continuous, its density c is, as usual, given by

$$c(u_1,\ldots,u_n) = \frac{\partial^n C(u_1,\ldots,u_n)}{\partial u_1 \ldots \partial u_n}. \tag{3}$$

In the empirical part of this paper we will use the Gaussian, and Joe-Clayton copulas. They are defined as follows:

$$C_\rho^{Gauss}(u,v) = \Phi_\rho\left(\Phi^{-1}(u), \Phi^{-1}(v)\right), \tag{4}$$

$$C_{\kappa,\gamma}^{J-C}(u,v) = 1 - \left(1 - ([1-(1-u)^{\kappa}]^{-\gamma} + [1-(1-v)^{\kappa}]^{-\gamma} - 1)^{-1/\gamma}\right)^{1/\kappa}. \quad (5)$$

In (4), Φ_{ρ} denotes the distribution of standard 2-dimensional normal vector with the linear correlation coefficient ρ, and Φ stands for the standard normal distribution function. The parameters in the Joe-Clayton copula (5) are assumed to satisfy the conditions: $\kappa \geq 1$, $\gamma \in [-1,\infty) \setminus \{0\}$. For $\kappa = 1$, the Joe-Clayton copula becomes the Clayton copula. In the limit case $\gamma = 0$, the Clayton copula approaches the independent copula (Nelsen, 2006).

The most popular measure of dependence, the usual Pearson linear correlation, is inappropriate and can be misleading outside the world of elliptical distributions. In such a situation it is better to use Spearman's rank correlation and Kendall's tau (Embrechts *et al*., 2002), which are copula-based dependence measures. If (X,Y) is a random vector and $(\tilde{X},\tilde{Y})$ is its independent copy then Kendall's tau of (X,Y) is defined as

$$\tau(X,Y) = P\{(X-\tilde{X})(Y-\tilde{Y}) > 0\} - P\{(X-\tilde{X})(Y-\tilde{Y}) < 0\}. \quad (6)$$

It can be expressed in terms of the copula C for a continuous vector (X,Y) as

$$\tau(X,Y) = 4\iint_{[0,1]^2} C(u,v)\mathrm{d}C(u,v) - 1. \quad (7)$$

For the Gaussian copula C_{ρ}^{Gauss}, Kendall's tau equals $\frac{2}{\pi}\arcsin(\rho)$. A direct formula for Kendall's tau for the Joe-Clayton copula $C_{\kappa,\gamma}^{\mathrm{J-C}}$ is not known to us, but we have derived the formula

$$\tau(\kappa,\gamma) = 1 + \frac{2}{\kappa\gamma} - \frac{4}{\kappa\gamma}\int_0^1 x^{1-\kappa}(1-(1-x^{\kappa})^{\gamma+1})dx \quad (8)$$

which allows for numerical computation.

If X and Y are random variables with distribution functions F and G then the coefficient of tail dependence, upper, λ_U, and lower, λ_L are defined as follows

$$\lambda_U = \lim_{q\to 1^-} P(Y > G^{-1}(q) \mid X > F^{-1}(q)), \quad (9)$$

$$\lambda_L = \lim_{q\to 0^+} P(Y \leq G^{-1}(q) \mid X \leq F^{-1}(q)), \quad (10)$$

provided that the limits exist. If they are greater than 0, then the variables are said to exhibit upper (lower) tail dependence. The coefficients dependent only on the copula of X and Y. For the Gaussian copula it holds $\lambda_U = \lambda_L = 0$ (Embrechts *et al*., 2002), meaning asymptotic independence in the tails. In the Joe-Clayton copula, $\lambda_U = 2 - 2^{1/\kappa}$ and $\lambda_L = 2^{-1/\gamma}$ for $\gamma > 0$ (Patton, 2006).

3. Markov-Switching Copula Models

The notion of conditional copula introduced by Patton (2004, 2006) allows to apply copulas for modeling the joint distribution r_t conditional on information set Ω_{t-1}, where $r_t = (r_{1,t}, r_{2,t})$ is a bivariate vector of financial returns. In this paper we consider the following general dynamic conditional copula model

$$r_{1,t} \mid \Omega_{t-1} \sim F_t(\cdot), \quad r_{2,t} \mid \Omega_{t-1} \sim G_t(\cdot), \tag{11}$$

$$r_t \mid \Omega_{t-1} \sim C_t(F_t(\cdot), G_t(\cdot) \mid \Omega_{t-1}), \tag{12}$$

where the set Ω_t includes the up to time t information on the returns on both considered financial assets, and C_t is the conditional copula joining the marginal conditional distributions. Further, we assume that

$$r_t = \mu_t + y_t, \quad \mu_t = E(r_t \mid \Omega_{t-1}), \tag{13}$$

$$y_{i,t} = \sigma_{i,t}\varepsilon_{i,t}, \quad \sigma_{i,t}^2 = \mathrm{var}(r_{i,t} \mid \Omega_{t-1}), \tag{14}$$

$$\varepsilon_{i,t} \sim IID\ Skew\_t(0,1,\xi_i,\eta_i), \quad \sigma_{i,t}^2 = \omega_i + \alpha_i y_{i,t-1}^2 + \beta_i \sigma_{i,t-1}^2, \tag{15}$$

where $Skew\_t(0,1,\xi,\eta)$ denotes the standardized skewed Student t distribution with $\eta > 2$ degrees of freedom, and skewness coefficient $\xi > 0$ (Lambert and Laurent, 2002).

In a Markov-switching copula model (MSC model) we use for modeling the conditional dependence between the financial returns, the joint conditional distribution has the form $r_t \mid \Omega_{t-1} \sim C_{S_t}(F_t(\cdot), G_t(\cdot) \mid \Omega_{t-1})$, where S_t is a homogeneous Markov chain with state space $\{1,2\}$. The parameters of the applied MSC model are the parameters of the univariate models for the marginal distributions (GARCH(1,1) with the standardized skewed Student's t distributions for the innovations), the parameters of the copulas C_1 and C_2, and the transition probabilities

$$p_{11} = P(S_t = 1 \mid S_{t-1} = 1), \quad p_{22} = P(S_t = 2 \mid S_{t-1} = 2). \tag{16}$$

The conditional probabilities $P(S_t = j \mid \Omega_{t-1})$, $j = 1, 2$, are calculated by means of Hamilton's filter:

$$P(S_t = j \mid \Omega_{t-1}) = \sum_{i=1}^{2} p_{ij} P(S_{t-1} = i \mid \Omega_{t-1}), \tag{17}$$

$$P(S_t = j \mid \Omega_t) = \frac{c_j(u_t \mid S_t = j, \Omega_{t-1}) P(S_t = j \mid \Omega_{t-1})}{\sum_{i=1}^{2} c_i(u_t \mid S_t = i, \Omega_{t-1}) P(S_t = i \mid \Omega_{t-1})}, \tag{18}$$

where $p_{12} = P(S_t = 2 \mid S_{t-1} = 1) = 1 - p_{11}$, $p_{21} = P(S_t = 1 \mid S_{t-1} = 2) = 1 - p_{22}$, $u_t = (u_{1,t}, u_{2,t})'$, $u_{1,t} = F_t(r_{1,t})$, $u_{2,t} = G_t(r_{2,t})$, and $c_j(\cdot \mid S_t = j, \Omega_{t-1})$ is the density of the conditional copula coupling the conditional marginal distributions in regime *j*. The maximized log-likelihood function was of the form

$$L = \sum_{t=1}^{T} \ln\left(\sum_{j=1}^{2} c_j(u_t \mid S_t = j, \Omega_{t-1}; \theta) P(S_t = j \mid \Omega_{t-1}; \theta)\right) + \\ + \sum_{t=1}^{T} \ln\left(f_t(r_{1,t} \mid \Omega_{t-1}; \theta_1)\right) + \sum_{t=1}^{T} \ln\left(g_t(r_{2,t} \mid \Omega_{t-1}; \theta_2)\right), \quad (19)$$

where f_t and g_t are the density functions corresponding to F_t and G_t, fitted using the AR-GARCH models.

4. The Data and Empirical Results

The data we use in this paper consist of daily returns on two Polish stock indices, WIG20 and MIDWIG. The indices represent, respectively, the largest and medium-sized companies listed on the Warsaw Stock Exchange. The sample period is from January 9, 2001 to March 16, 2007. The daily return series r_t is defined as $r_t = 100(\ln P_t - \ln P_{t-1})$, where P_t is the closing quotation on day *t*. Table 1 presents descriptive statistics for the return series.

Table 1. Descriptive statistics for the return series (January 9, 2001 – March 16, 2007)

Index	Mean	Std. Dev.	Minimum	Maximum	Skewness	Kurtosis
WIG20	0.0412	1.4583	-5.7306	5.4830	0.0412	4.0421
MIDWIG	0.0966	0.8731	-5.6449	4.1011	-0.56426	6.3457

For the MIDWIG index kurtosis is much higher than for the WIG20, and the same holds for the absolute value of skewness which is negative in the case of the MIDWIG and positive for the WIG20. The return series showed some autocorrelation and in all cases conditional homoskedasticity was strongly rejected by the Engle test.

We estimated the MSC model applying a two-stage estimation method. Taking into account the obtained values of the information criteria and the results of performed likelihood ratio tests, we assumed that the dependence structure in one of the regimes was governed by the Gaussian copula C_ρ^{Gauss}. We decided to mark this a regime by 1. Regime 2 was chosen to be governed by the Joe-Clayton copula $C_{\kappa,\gamma}^{\mathrm{J-C}}$. For the sake of place, we report only the bivariate estimation results. They are presented in Table 2.

Table 2. Parameter estimates for the MSC model (standard errors in parentheses)

	ρ	p_{11}	p_{22}	κ	γ
WIG20&MIDWIG	0.7522 (0.022)	0.9555 (0.0361)	0.9152 (0.0797)	1.1997 (0.1001)	1.0197 (0.1872)

The obtained estimates of the transition probabilities contain additional information which is of importance from the practical point of view. Namely, we can derive from them the unconditional probabilities $P(S_t = i)$ of being the process in regime i, the inverses $mtr(i)$ of which can be interpreted as expected times of return of the process to regime i, and the expected duration $d(i)$ of regime i, $i = 1,2$. The corresponding formulas are as follows (cf. Durrett, 1999):

$$d(i) = \frac{1}{1 - p_{ii}}, \tag{20}$$

$$P(S_t = 1) = \frac{1 - p_{22}}{2 - p_{11} - p_{22}}, \quad P(S_t = 2) = \frac{1 - p_{11}}{2 - p_{11} - p_{22}}. \tag{21}$$

Table 3 contains the estimates for these quantities as well as Kendall's tau coefficients for the estimated copulas. The estimates for upper and lower tail coefficients for the Joe-Clayton copula together with the standard errors calculated by the delta method are presented in Table 4.

Table 3. Kendall's tau for the estimated copulas, the unconditional probability of regime 1, mean times of return to the regimes, and durations for the regimes

τ^{Gauss}	τ^{J-C}	$P(S_t = 1)$	$mtr(1)$	$mtr(2)$	$d(1)$	$d(2)$
0.5420	0.3786	0.6558	1.5248	2.9055	22.4624	11.7892

Table 4. Estimates for upper and lower tail coefficients (standard errors in parentheses)

λ_U	λ_L
0.2179 (0.0859)	0.5067 (0.0632)

From the obtained results it follows that regime 1, governed by the Gaussian copula, is more stable, and, on average, it lasts about twice as long as the one governed by the Joe-Clayton copula. The dependence between the indices returns in regime 1, measured by Kendall's tau, is almost one and a half as strong as in the other regime. Regime 2 is, however, characterized by significant tail dependence of asymmetric type. Significantly greater coefficient of lower tails dependence means that the dependencies between returns on the WIG20 and MIDWIG are much stronger during bear markets than in bull markets. In order to compare our results with ones available with more traditional tools, we also estimated the conditional dependencies between the return series using Engle's (2002) DCC model with Student's t errors. Figure 1 shows the dynamics of the conditional Kendall's tau estimated with this model and with the MSC model. It should be noticed that the *t*-DCC model gives higher estimates. Figure 2 depicts the dynamics of the conditional upper and lower tail coefficients estimated by means of the MSC model.

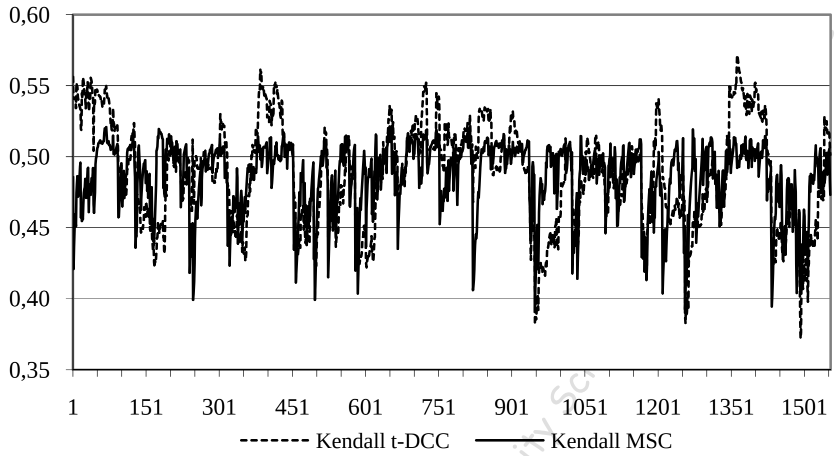

Figure 1. The dynamics of the conditional Kendall's tau coefficients estimated with the *t*-DCC and MSC models

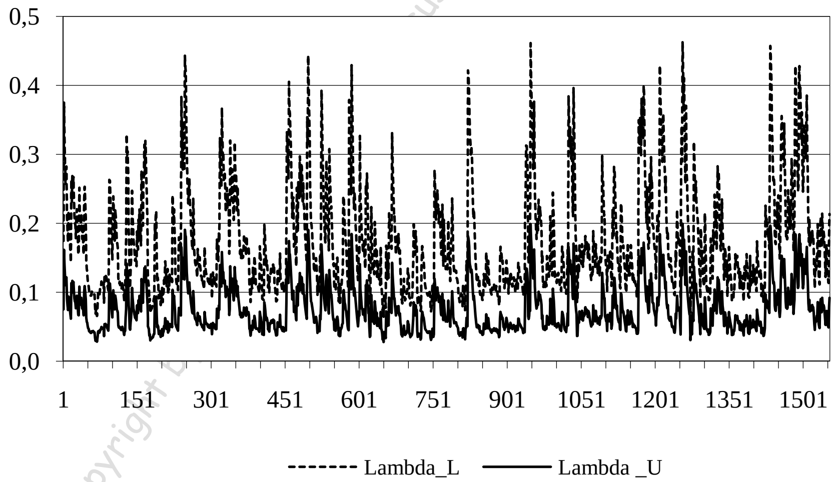

Figure 2. The dynamics of the conditional coefficients of lower and upper tail dependence estimated with the MSC model

5. Conclusions

In this paper, using a Markov-switching copula model, we investigated conditional dependencies between daily returns on two Polish stock indices, WIG20 and MIDWIG, representing the largest and medium-sized companies listed on the Warsaw Stock Exchange. We were especially interested in discovering patterns of dependence that cannot be reproduced by traditional multidimensional volatility models with elliptical conditional distributions, including asymmetry in tail dependence. To this end, we applied a Markov-switching copula model that allowed us to model the dynamics of such copula-based dependence measures between the returns as Kendall's tau and tail dependence coefficients. We identified two regimes in the dependence structure: the first more stable, governed by the Gaussian copula, and the second with the expected duration about half as much as that of the first one, exhibiting significant asymmetry in tail dependence, described by the Joe-Clayton copula. Our findings that the returns on the indices are much stronger dependent during bear market than during bull market, and that the conditional dependence measured by the Kendall's tau calculated from the Markov-switching copula model is weaker than the one obtained from the Student-*t*-DCC model can be of importance for risk management.

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