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Bayesian Forecasting of the Discounted Payoff of Options on WIG20 Index under Stochastic Volatility and Stochastic Interest Rate*

1. Introduction

The classical Black-Scholes model assumes that asset returns follow a continuous diffusion process with constant conditional volatility and constant interest rate. Thus, numerous studies on option pricing have relaxed the unrealistic assumptions. Firstly, the assumption of constant interest rate was relaxed to allow for stochastic interest rate, as in Merton (1973) or Turnbull and Milne (1991). Secondly, the assumption of constant volatility was relaxed. Hull and White (1987) was one of the first papers to derive an option pricing formula for a European call option in stochastic volatility model. In their model, the interest rate is constant and the conditional variance is uncorrelated with the asset price. Heston (1993) presented a close-form solution for options on assets with stochastic volatility, constant interest rate, and a non-zero correlation between volatility and asset prices. The option pricing model incorporating both stochastic interest rates and stochastic volatility is not often considered in literature. Amin and Ng (1993) built the option pricing model which incorporates both a stochastic interest rate and a stochastic volatility process for stock returns.

The aim of the paper is to check whether allowing interest rates to be stochastic improves forecasting performance of the discounted payoff of options on the WIG20 index. We compare the results obtained in the option pricing model under stochastic volatility and stochastic interest rate (allowing the interest rate to follow an SV process) with those in constant interest rate model (an univariate SV model for the underlying asset).

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The structure of the article is as follows: section 2 consists of a short presentation of the Bayesian univariate SV model with correlated errors, section 3 includes a brief presentation of the Bayesian bivariate SV model, section 4 focuses on the Bayesian forecasting of the discounted payoff of a European call option, section 5 presents the posterior results connected with options on the WIG20 index, and finally, section 6 incorporates the conclusions.

2. Bayesian Univariate AR(1)-CSV Model

Let x_t denote the price of the underlying asset at time t, $t = 1, 2, ..., T+s$. The growth rate y_t is defined as $y_t = 100 \ln (x_t/x_{t-1})$ and modelled using the discrete-time correlated SV model (CSV) considered by Jacquier, Polson and Rossi (2004). The CSV model specifies a log-normal autoregressive process for the conditional variance with correlated innovations in the conditional mean and conditional variance equations. The univariate AR(1) - CSV model is defined as follows:

$$y_t = \delta_1 + \rho_1 (y_{t-1} - \delta_1) + \varepsilon_t , \tag{1}$$

$$\varepsilon_t = u_t \sqrt{h_t} , \quad \ln h_t = \gamma + \phi \ln h_{t-1} + \sigma_h \eta_t , \tag{2}$$

where $(u_t , \eta_t)' \sim iiN\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \right)$, $t = 1, 2, \ldots, T+s$,

iiN denotes independent, identically and normally distributed. One interpretation for the latent variable h_t is that it represents the random and autocorrelated flow of new information into financial markets (see Clark, 1973). Here ϕ is connected with the volatility persistence, σ_h is the volatility of the log-volatility. The above model can pick up the kind of asymmetric behaviour often observed in stock price movements, which is known as the leverage effect when the correlation ρ is negative[1]. Also, the negative value of ρ induces left-skewness in the marginal distribution of ε_t. In order to complete the Bayesian model, we have to specify a prior distribution on the parameter space. In this paper we use the following prior structure:

$$p(\delta_1, \rho_1, \gamma, \phi, \sigma_h^2, \rho) = p(\delta_1) p(\rho_1) p(\gamma) p(\phi) p(\sigma_h^2, \rho) ,$$

where we use proper prior densities of the following distributions: $\delta_1 \sim N(0, 1)$, $\rho_1 \sim U(-1,1)$, $\gamma \sim N(0, 10^2)$, $\phi \sim N(0, 10^2)\, I_{(-1,1)}(\phi)$, $\tau \sim IG(1, 0.005)$, $\psi | \tau \sim N(0, \tau/2)$, $\psi = \sigma_h \rho$, $\tau = \sigma_h^2 (1 - \rho^2)$ (see Jacquier, Polson and Rossi, 2004).

[1] If ρ is negative, then a negative innovation u_t is associated with higher contemporaneous and subsequent volatilities. On the other hand, a positive innovation u_t is associated with a decrease in the volatility (see Jacquier, Polson and Rossi, 2004).

The prior distribution for δ_1 is standardized normal, U(-1,1) denotes the uniform distribution over (-1,1). The prior distribution for ϕ is normal, truncated by the restriction that the absolute value of ϕ is less than one ($I_{(-1,\,1)}(.)$ denotes the indicator function of the interval (-1, 1), which is the region of stationarity of $\ln h_t$). The symbol $IG(v_0, s_0)$ denotes the inverse Gamma distribution with the mean $s_0/(v_0-1)$ and the variance $s_0^2/[(v_0-1)^2(v_0-2)]$ (thus, when $\rho = 0$, the prior mean for σ_h^2 does not exist, but σ_h^{-2} has a Gamma prior with the mean 200 and standard deviation 200). The initial condition h_0 is equal to y_0^2. These assumptions reflect rather weak prior knowledge about the parameters.

3. Bayesian Bivariate VAR(1)-TSV Model

Let now $x_{j,t}$ denote the price of asset j at time t for $j = 1, 2$ and $t = 1, 2, ..., T$ (in this paper $x_{1,t}$ and $x_{2,t}$ are respectively the index level and interest rate at time t). The vector of growth rates $y_t = (y_{1,t}, y_{2,t})'$, each defined by the formula $y_{j,t} = 100 \ln (x_{j,t}/x_{j,t-1})$, is modelled using the basic VAR(1) framework:

$$y_t - \delta = R(y_{t-1} - \delta) + \xi_t, \quad t = 1, 2, \dots, T+s, \tag{3}$$

where T observations are used in estimation, s is the forecasting horizon. In (3) δ is a 2-dimensional vector, R is a 2×2 matrix of parameters, and ξ_t is a bivariate SV process. We assume that, conditionally on the latent variable vector Ω_t, ξ_t follows a bivariate Gaussian distribution with the mean vector $0_{[2\times1]}$ and the covariance matrix Σ_t, i.e. $\xi_t \mid \Omega_t \sim N(0_{[2\times1]}, \Sigma_t)$, $t = 1, 2, ..., T+s$. For the matrix Σ_t the Cholesky decomposition is used (see Tsay, 2002):

$$\Sigma_t = L_t\, G_t\, L_t', \tag{4}$$

where L_t is a lower triangular matrix with unitary diagonal elements, G_t is a diagonal matrix with positive diagonal elements:

$$L_t = \begin{bmatrix} 1 & 0 \\ q_{21,t} & 1 \end{bmatrix} G_t = \begin{bmatrix} q_{11,t} & 0 \\ 0 & q_{22,t} \end{bmatrix}.$$

$\{q_{21,t}\}$, and $\{\ln q_{jj,t}\}$ ($j = 1,2$), as in the univariate SV specification, are standard univariate autoregressive processes of order one, namely

$$\ln q_{11,t} - \gamma_{11} = \phi_{11}(\ln q_{11,t-1} - \gamma_{11}) + \sigma_{11}\eta_{11,t},$$

$$\ln q_{22,t} - \gamma_{22} = \phi_{22}(\ln q_{22,t-1} - \gamma_{22}) + \sigma_{22}\eta_{22,t}, \quad q_{21,t} - \gamma_{21} = \phi_{21}(q_{21,t-1} - \gamma_{21}) + \sigma_{21}\eta_{21,t},$$

where $\eta_t = (\eta_{11,t}, \eta_{21,t}, \eta_{22,t})'$ and $\eta_t \sim iiN(0_{[3\times1]}, I_3)$, $\Omega_t = (q_{11,t}, q_{22,t}, q_{21,t})'$.

The Cholesky decomposition of Σ_t requires no parameter constraints for the positive definiteness of Σ_t. The matrix Σ_t is positive definite if $q_{jj,t} > 0$ for $j = 1$, 2, which is achieved by modelling $\ln q_{jj,t}$ instead of $q_{jj,t}$. If $|\phi_{ij}| < 1$ (i, j=1, 2,

$i \geq j$), then $\{\ln q_{11,t}\}$, $\{\ln q_{22,t}\}$, and $\{q_{21,t}\}$ are stationary and the SV process is a white noise (see Pajor, 2005). We make similar assumptions about the prior distributions as previously. In particular: $\gamma_{ij} \sim N(0, 10^2)$, $\phi_{ij} \sim N(0, 10^2)I_{(-1,1)}(\phi_{ij})$, $\sigma_{ij}^2 \sim IG(1, 0.005)$, $\ln q_{ii,0} \sim N(0, 10^2)$ $i, j \in \{1, 2\}$, $i \geq j$, $q_{21,0} \sim N(0, 10^2)$. For δ and R we assume the multivariate standardised Normal prior $N(0, I_6)$, truncated by the restriction that all eigenvalues of R lie inside the unit circle (similar to Osiewalski and Pipień, 2004). The prior distributions are assumed to be independent of each other.

4. Application to Bayesian Forecasting of the Discounted Payoff

An important application of the stochastic volatility models is the option pricing. The payoff at time $T+s$ of a European call option is given by

$$V_{T+s} = \max(x_{T+s} - K, 0), \tag{5}$$

where K is the exercise price (strike price), x_{T+s} is the price of the underlying asset at time $T+s$ (no dividend being paid), s is the time to maturity. The present value of payoff considered at time T under stochastic interest rate is[2]

$$W_{T|T+s} = \exp\left(- \int_T^{T+s} r_t \, dt \right) \max(x_{T+s} - K, 0), \tag{6}$$

where r_t is the interest rate at time t. This discounted payoff is a random variable as a measurable function of x_{T+s} and r_t, $t \in [T, T+s]$, which are random. The distribution of $W_{T|T+s}$ is induced by the predictive distributions of x_{T+s} and r_t, $t \in [T, T+s]$. The Bayesian approach naturally provides a tool to compute the predictive distribution of the discounted payoff, $W_{T+s|T}$ (see Bauwens and Lubrano, 1998; Osiewalski and Pipień, 2003). The predictive density of the payoff is defined by

$$p(W_{T|T+s} \mid y) = \int p(W_{T|T+s} \mid \theta, y) \, p(\theta \mid y) \, d\theta,$$

where y is the sample of returns used for estimation, $p(\theta \mid y)$ is the posterior density of the parameters and latent variables of the Bayesian econometric model.

It is important to stress that the specification (1) relaxes the Black and Scholes constant volatility assumption. In this case the volatility follows a separate process. The specification (3) relaxes the Black and Scholes constant volatility and constant interest rate assumptions, furthermore allows the interest rate to follow an SV process. In the univariate AR(1)-CSV model the stock price involves two sources of the risk: the stock return risk and volatility risk. Thus, the investor incurs the risk from a randomly evolving asset price and the risk of a

[2] In a discrete-time model the integral in Equation (6) is replaced by the summation.

randomly evolving volatility. Note that in the bivariate VAR(1)-TSV model (presented above) there are four sources of the risk: the risk from the asset price, the volatility of the underlying asset, the interest rate, and from the volatility of the interest rate. It is clear that this model is incomplete. It is well known that in an incomplete market there is no unique fair price and no universal pricing algorithm. There are several alternative methodologies which have been proposed as pricing mechanisms (see Hobson, 2004 for a review of the methods). In this paper we consider the original probability measure (the physical measure). This means that we assume that both the stochastic interest rate volatility and stochastic interest rate, as well as the stochastic asset return volatility have zero risk premium. The Bayesian approach takes completely into account the uncertainty, which come from prediction and from the parameters, by construction of the predictive distribution of the discounted payoff. The predictive option price may be defined as the median of $W_{T|T+s}$ (see Osiewalski and Pipień, 2003).

5. Empirical Results

We use daily observations (closing quotes) of the WIG20 index and WIBOR6m (the 6- month Warsaw Interbank Offered Rate) over the period from January 2, 2001 to December 31, 2004. The data was downloaded from www.money.pl. The dataset of the daily logarithmic growth rates (expressed in percentage points) y_t consists of 1005 observations (for each series). The first observation is used to construct initial conditions, thus $T = 1004$ (the number of modelled observations). We consider all European call options on the WIG20 index, which were quoted on Warsaw Stock Exchange (WSE) on December 31, 2004 (at the end of the observed sample). The exercise dates are March 18, 2005 (i.e. $s = 55$ trading days) or June 17, 2005 (i.e. $s = 115$ trading days). As the proxy for the unobservable short rate, the 6 month WIBOR rate is used. As justified Jiang (1998), and Jiang and Sluis (1999) the use of the 6 month WIBOR rate is a compromise between an instantaneous rate (overnight rates) and avoiding some of the associated spurious microstructure effects. In the VAR(1)-TSV model the first component of the vector y_t is the growth rate of the WIG20 index, the second one is the growth rate of the WIBOR6m.

We report in Table 1 the main characteristics of the predictive distributions of the discounted payoff for the European call option on the WIG20 index. In the univariate AR(1)-CSV model (with constant interest rate), according to the recommendation of the Warsaw Stock Exchange and the Polish National Depository for Securities it was assumed that the risk-free interest rate is 6,5% per annum (i.e. r = 6,5% on annual base, see Kostrzewski and Pajor, 2007) [3].

[3] The correlation between index returns and volatility in the AR(1)-CSV model is insignificantly different from zero. The posterior mean of ρ in the AR(1)-CSV model is equal to -0.0018 with the standard deviation 0.1611. Thus, formal Bayesian testing (not

Table 1. The predictive characteristics of the discounted payoff. Here $W = W_{T|T+s}$

Model		s=55					s=115			
CSV	quantile of order:	1700	1800	1900	2000	2100	1800	1900	2000	2100
	0.05	22.63	0	0	0	0	0	0	0	0
	0.25	185.07	86.49	0	0	0	63.24	0	0	0
	0.50	**298.84**	**200.26**	**101.68**	**3.1**	**0**	**243.04**	**146.01**	**48.98**	**0**
	0.75	419.74	321.16	222.58	124	25.11	441.13	344.1	247.07	150.04
	0.95	620.00	521.42	422.84	324.26	225.68	785.85	688.82	591.79	494.76
	IQR	**234.67**	**234.67**	**222.58**	**124.00**	**25.11**	**377.89**	**344.10**	**247.07**	**150.04**
	Pr(W=0|y)[4]	0.038	0.117	0.274	0.493	0.706	0.179	0.293	0.429	0.568
true value of discounted payoff		271.11	172.53	73.94	0	0	217.43	120.36	23.30	0
TSV	quantile of order:	1700	1800	1900	2000	2100	1800	1900	2000	2100
	0.05	47.43	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	0.25	198.71	100.13	1.55	0.00	0.00	90.21	0.00	0.00	0.00
	0.50	**304.42**	**205.84**	**107.26**	**8.68**	**0.00**	**254.82**	**157.79**	**60.45**	**0.00**
	0.75	415.40	316.82	218.24	119.66	21.08	433.38	336.04	239.01	141.67
	0.95	597.99	499.41	400.83	302.25	203.67	743.07	645.73	548.70	451.36
	IQR	**216.69**	**216.69**	**216.69**	**119.66**	**21.08**	**343.17**	**336.04**	**239.01**	**141.67**
	Pr(W=0|y)	0.026	0.094	0.247	0.478	0.709	0.146	0.260	0.404	0.558
true value of discounted payoff		271.07	172.50	73.93	0	0	217.79	120.56	23.33	0
Quotations on December 31,2004		285	200	105	50	15.1	189	119	78	45

In both models the true values of the discounted payoff are located between the median and the quantile of order 0.75 or between the quantile of order 0.25 and the median, but in the close neighbourhood of the medians. Also, the observed market prices of the options are closed to the medians. It is worth to stress, that the inter-quartile range (IQR) indicates huge uncertainty of the future payoff. In Figure 1 we present histograms of the predictive distribution of the discounted payoff of the European call option with the exercise price K equals 1800 index's points and s = 55. The first bars of graphs denote probability of non-exercise of the option. The little red points represent the true values of the discounted payoff. They are located between the first quartile and the median of the predictive distributions of the discounted payoff. The predictive histograms are characterized by huge dispersion and thick tails, thus uncertainty about the future value of payoff was very big ex-ante. In the last column in Table 2 we have the aver-

presented here) does not lead to rejection of the hypothesis of a zero correlation between volatility and index returns.

[4] The settlement prices for derivative securities were equal to 1975 (for s = 55) and 2024 (for s = 115).

age (mean) forecasting errors[5] (MFE). The level of MFE in the bivariate VAR(1)-TSV model (with stochastic interest rates) is higher than in the univariate AR(1)-CSV model (with constant interest rate). Thus the VAR(1)-TSV model performs worse than the AR(1)-CSV model. The empirical results allow us to infer that stochastic interest rates may not be important for the forecasting of the discounted payoff. It seems that stochastic interest rate has minimal impact on option prices. Surprisingly, the uncertainty of the future value of the payoff (measured by IQR) is bigger in the univariate AR(1)-CSV model.

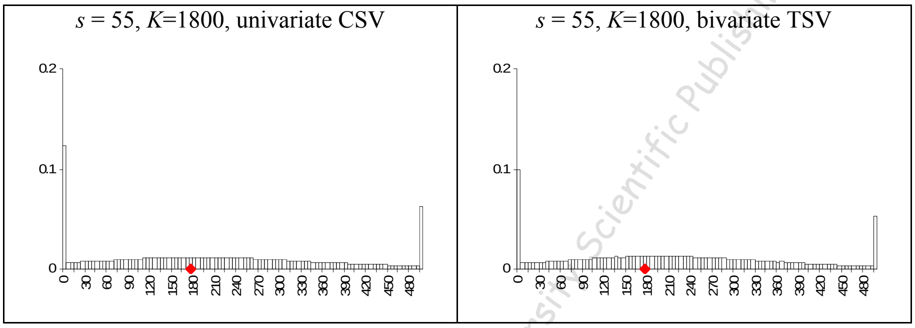

Figure 1. Histograms of the predictive distributions of the discounted payoff

Table 2. The predictive median of $W_{T|T+s}$ minus the true value of the discounted payoff

Model	s=55					s=115				
	1700	1800	1900	2000	2100	1800	1900	2000	2100	**MFE**
CSV	27.73	27.73	27.74	3.10	0.00	25.61	25.65	25.68	0.00	**18.14**
TSV	33.25	33.28	33.31	8.68	0.00	36.82	37.11	37.09	0.00	**24.39**

6. Conclusions

In this paper the bivariate Stochastic Volatility models (with stochastic volatility and stochastic interest rate) and the univariate Correlated Stochastic Volatility model (with stochastic volatility and constant interest rate) are used in Bayesian forecasting of the payoff of the European call options on the WIG20 index. The empirical results indicate that allowing interest rates to be stochastic does not improve forecasting performance of the discounted payoff. The true values of the discounted payoff (observed ex-post) are located between the first quartile and the median of the predictive distribution of the discounted payoff, but the predictive distributions of the discounted payoff have such huge dispersion that they are hardly informative for the purpose of option pricing.

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[5] The average forecasting error is defined as: $MFE = (1/n)\sum_{i=1}^{n} \hat{C}_i - C_i$, where n is the number of options used in the comparison, C_i and $\hat{C}_i$ represents the true value of the discounted payoff and the predictive median of the discounted payoff, respectively.

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