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Witold Orzeszko
Nicolaus Copernicus University in Toruń

Properties of STUR Processes in the Framework of Chaos Theory[1]

1. Introduction

Stochastic unit-root processes – STUR have a root varying around unity, so they can be stationary for some periods and nonstationary for the others. Since properties of STUR processes make them quite natural in modeling of economic time series, an appropriate identification method of stochastic unit-roots is required. However, standard tests for a unit root can not distinguish between STUR processes and integrated ones. The aim of this paper is to examine if chaos theory provides methods of time series analysis which can be used to stochastic unit-roots identification.

This paper is organized as follows. Section 1 contains the introduction. Stochastic unit-root processes are defined in section 2. Two methods from chaos theory – the estimation of the largest Lyapunov exponent and R/S analysis are described in sections 3 and 4. The results of experiments verifying the usefulness of these methods to STUR identification are presented in section 5.

2. STUR Processes

Granger and Swanson (1997) defined stochastic unit-root processes as:

$$x_t = a_t x_{t-1} + \varepsilon_t , \tag{1}$$

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where ε_t is zero mean, i.i.d. with variance σ_ε^2, and where $a_t = \exp(\alpha_t)$ with α_t a Gaussian stationary series having mean *m*, variance σ_α^2 and power spectrum $g_\alpha(\omega)$.

A particular case where α_t is an AR(1) process is considered in this paper, i.e.:

$$\alpha_t = \mu + \rho\alpha_{t-1} + \eta_t, \tag{2}$$

where $|\rho| < 1$ and $\eta_t \sim N(0;\sigma_\eta^2)$ is i.i.d. independent of ε_t.

STUR processes are a specific subclass of time-varying processes. On the other hand, they can be considered as a generalization of unit-root processes, since Equation 1 gives the random walk process if $\alpha_t \equiv 0$. However, in a general case roots of STUR processes vary around unity, which makes them stationary for some periods and nonstationary for the others.

An application of STUR processes in modeling of economic time series must be preceded by an identification of stochastic unit-roots. Granger and Swanson (1997) showed that one of the most popular methods of unit-roots identification – augmented Dickey-Fuller (ADF) test, can not distinguish between time series generated by STUR and random walk models. Leybourne, McCabe and Tremayne (1996) proposed the test, which identifies stochastic unit-root processes much better than ADF. However, the power of this test depends on the length of investigated time series and on the variance σ_η^2 (Granger, Swanson (1997)).

3. R/S Analysis

The rescaled range, or R/S analysis detects long-memory processes so it is used to identify non-random time series. The calculation of the rescaled range for a given time series y_t, where $t = 1,2,...,n$, consists of the following steps (see Peters (1994)):

1. the mean $M^{(n)}$ and the standard deviation $S^{(n)}$ of y_t are calculated,
2. the time series $Y_k^{(n)} = \sum_{t=1}^{k}(y_t - M^{(n)}),\ k = 1,2,...,n$, is determined,
3. the range $R^{(n)} = \max_k(Y_k^{(n)}) - \min_k(Y_k^{(n)})$ is computed,
4. the rescaled range $R/S_n \overset{df}{=} R^{(n)}/S^{(n)}$ is calculated.

However, it is the Hurst exponent which is the final result of R/S analysis. To estimate its value, an investigated time series x_t, $t = 1,2,...,N$, must be divided into separable subsequences of the length n, where the number n is, in turn, each of the dividers of the number N, satisfying the condition $2 \leq n \leq N/2$.[2] For the fixed n, the values $R/S_n^{(i)}$ are calculated for the each i-th subsequence, following steps 1–4. Next, the average $\overline{R/S_n^{(i)}}$ over all subsequences is computed and this value is set as R/S_n. Applying this procedure for each value n a sequence (R/S_n) is obtained. The essence of R/S analysis is that R/S_n follows the power law:

$$R/S_n = a \cdot n^{h(N)}, \tag{3}$$

or equivalently:

$$\ln(R/S_n) = h(N)\ln n + \ln a, \tag{4}$$

where the number $h(N)$ is called the Hurst exponent and a is a constant. Therefore, the Hurst exponent is estimated as a slope of the regression Equation 4.

The Hurst exponent equal to 0.5 indicates no long-term dependencies in the time series. However, it should be emphasized that 0.5 is the expected value of the Hurst exponent for short-memory time series of the infinite length. In the case of a finite number of observations, the expected value of $h(N)$ differ from 0.5. Therefore, in practice, the estimated Hurst exponent should be compared with the expected value for a short-memory time series of the same length. To this end, one of the approximating formula of calculating the value of $E(R/S_n)$ must be applied to. One of such formulas is the Anis and Lloyd's (1976) equation:

$$E(R/S_n) = \frac{\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{\pi}\cdot\Gamma\left(\frac{n}{2}\right)} \cdot \sum_{i=1}^{n-1}\sqrt{\frac{n-i}{i}}, \tag{5}$$

where $\Gamma(z)$ is the gamma function, defined as follows:

$$\Gamma(z) = (z-1)!, \quad \Gamma\left(z + \tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2^z}\cdot(2z-1)!!. \tag{6}$$

[2] To improve stability of estimates bigger values of n should be considered, e.g. $n \geq 10$ (Peters (1994), p. 63).

4. The Largest Lyapunov Exponent

Lyapunov exponents measure an average rate of a divergence/convergence of initially close states of dynamical systems. They quantify system's sensitivity to initial conditions, which is a crucial condition of chaotic systems. A presence of a positive exponent means that initially close trajectories of the system diverge exponentially. A larger positive Lyapunov exponent indicates a higher level of system's sensitivity, therefore a bigger strength of chaos.

In this paper, in order to estimate the largest Lyapunov exponent from a time series, the algorithm proposed independently by Kantz (1994) and Rosenstein et al. (1993) was applied. The algorithm consists of the following steps (see Kantz, Schreiber (1997)):

1) for each delay vector $\hat{x}_i^m = (x_i, x_{i-lag}, \ldots, x_{i-(m-1)lag})$, $i = (m-1)lag + 1, \ldots, N$, the set O_i consisting of the k nearest (in a sense of any fixed distance) neighbours $\hat{x}_{i_j}^m$ of $\hat{x}_i^m$ is determined[3],

2) for each $i = (m-1)lag + 1, \ldots, N$, and $n = 1, \ldots, n_{\max}$ the average

$$d_n(i) = \frac{1}{k} \sum_{\hat{x}_{i_j}^m \in O_i} \left| x_{i+n} - x_{i_j+n} \right| \tag{7}$$

is computed, where $n_{\max}$ is *a priori* set a natural number indicating the amount of iterations, while the divergence of states is analyzed,

3) the average d_n over all delay vectors is computed:

$$d_n = \frac{1}{N\text{-}(m\text{-}1)lag} \sum_{i=(m-1)lag+1}^{N} d_n(i), \tag{8}$$

4) the largest Lyapunov exponent is estimated as a slope of the regression equation:

$$\ln(d_n) = \ln(d_0) + \lambda n, \text{ for } n \geq 1. \tag{9}$$

It is clearly seen, that the results of this algorithm depends on the distance used to determine the nearest neigbours and on the parameters *m*, *lag*, *k* and $n_{\max}$ which are set *a priori* (see e.g. Orzeszko (2003)).

5. Experimental Results

In this section two methods from chaos theory – an estimation of the largest Lyapunov exponent and R/S analysis are considered. Their usefulness to identi-

[3] The parameters *m* and *lag* assume natural values and they are called, respectively, an embedding dimension and a time delay.

fication of stochastic unit-root is examined. The STUR process considered by Granger and Swanson (1997) is used in the simulations. On the ground of this process Granger and Swanson showed that augmented Dickey-Fuller test, cannot easily distinguish between exact and stochastic unit roots.

The investigated STUR process is defined by Equations 1 and 2 for $\rho = 0.6$, $\mu = -0.00003125$.[4] The two error processes $\varepsilon_t \sim N(0; 1)$ and $\eta_t \sim N(0; 0.0001)$ were simulated as independent, from the pseudorandom generator. 1000 time series of length 250 were generated from the considered STUR process and next 1000 time series of the same length, for comparison, from the random walk (RW) model $x_t = x_{t-1} + \varepsilon_t$.

First, to apply R/S analysis, time series of the first differences $\Delta x_t = x_t - x_{t-1}$ were determined. Since the number 249 has only 4 dividers, to improve the results of the Hurst exponent estimates, the time series of the differences were shortened to the first 240 observations (see Peters (1994)). The obtained frequency distributions of Hurst exponents are shown in Figure 1 and the calculated parameters of these distributions are summarized in Table 1. The Hurst exponent $E(h(240))$ for an independent process, calculated from the Anis and Lloyd formula equals 0.5760.

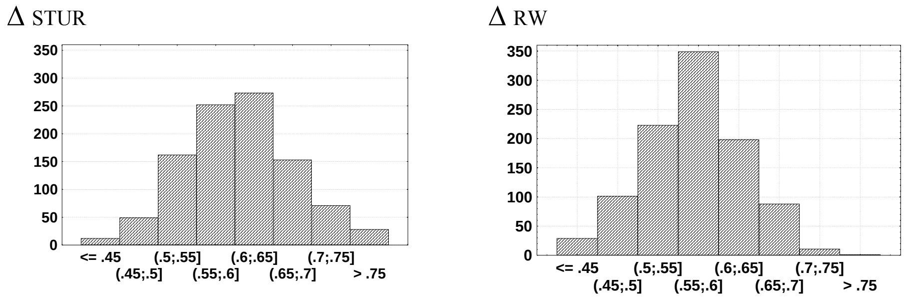
The Jarque-Bera test did not reject the null hypothesis of the normal distribution of the Hurst exponent for the Δ STUR process. Therefore, to verify the null $H_0 : E(h) = 0.5760$ against the alternative $H_1 : E(h) > 0.5760$, *Z*-statistic was applied. Since $Z = \frac{0.6057 - 0.5760}{0.0716} \cdot \sqrt{1000} = 13.117$, the null hypothesis is strongly rejected. This result indicates that, on an average, Hurst exponents of Δ STUR are bigger than for Δ RW, which means that R/S analysis may distinguish time series generated from these two processes.

Additionally, the Kolmogorov-Smirnov test was applied to verify a consistency of the distributions of Hurst exponents for Δ STUR and Δ RW processes[5]. The calculated Kolmogorov-Smirnov statistic KS equals 5.277, which means that the null hypothesis is rejected for the significance levels $\alpha = 0.05$ and $\alpha = 0.01$ as well. This result is an additional confirmation of the differences between the distributions of Hurst exponents for both analyzed processes.

[4] For the fixed ρ the value μ is determined to satisfy the condition $E(a_t) = 1$.

[5] The Kolmogorov-Smirnov statistic are usually noted as λ. The same symbol usually denotes Lyapunov exponents. To avoid confusing these quantities, in this paper, the Kolmogorov-Smirnov statistic is marked as *KS*.

Fig. 1. Frequency distributions of Hurst exponents for Δ STUR and Δ RW processes[6]



Source: Author's calculation.

Table 1. Parameters of the frequency distributions of Hurst exponents

	Δ STUR	Δ RW
Mean $\bar{h}$	0.6057	0.5700
Median	0.6038	0.5721
Maximum	0.8088	0.7764
Minimum	0.4054	0.3219
Standard deviation	0.0716	0.0618
Jarque-Bera (probability)	3.404 (0.182)	15.203 (0.001)

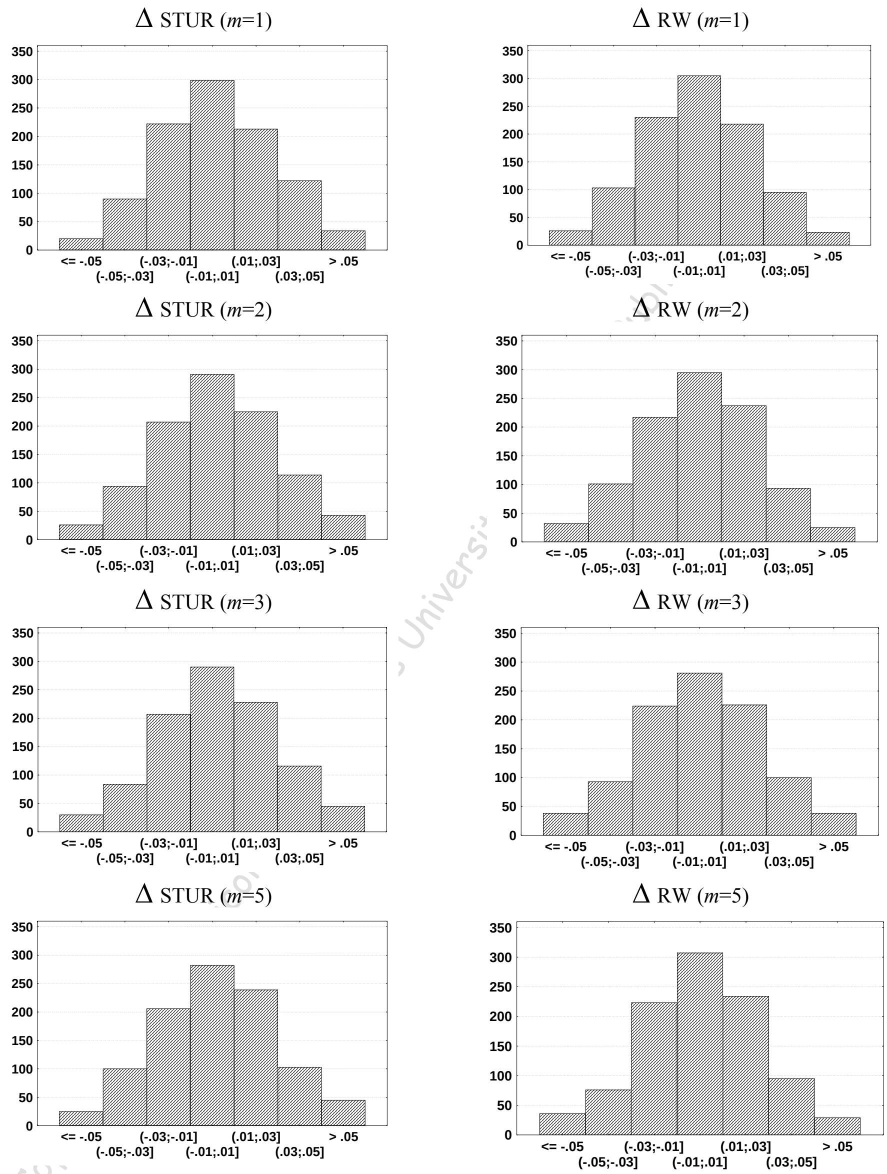
Source: Author's calculation.

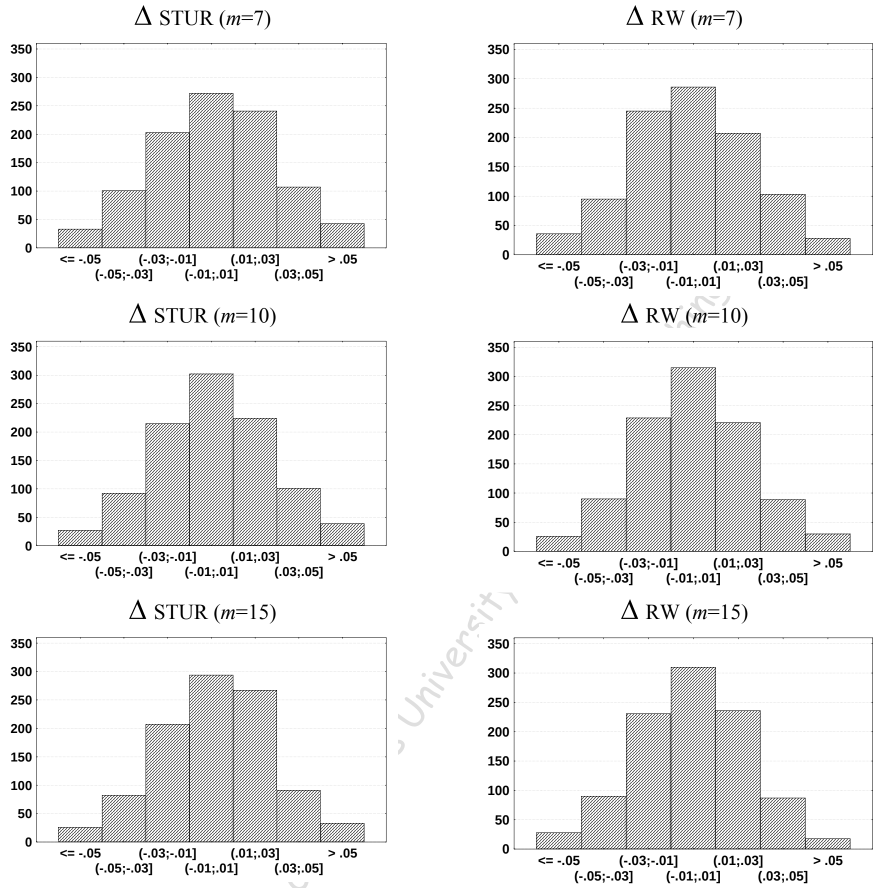
An estimation of the largest Lyapunov exponent was the second applied method. Since nonstationarity may disturb a reliability of chaos identification (see Kantz, Schreiber (1997)), the time series of the first differences Δx_t, $t = 2, 3, ..., 250$, were investigated. The aim of this research was to verify if STUR processes are sensitive to initial conditions and, if the estimation of the largest Lyapunov exponent may be a useful tool of distinguishing time series generated by STUR and random walk processes.

In computations the following values of the parameters were considered: $lag = 1$, $k = 1$, $n_{\max} = 5$ and $m = 1, 2, 3, 5, 7, 10, 15$. The obtained frequency distributions of the largest Lyapunov exponents are presented in Figure 2 and the calculated parameters of these distributions are summarized in Table 2.

[6] Of course in the case of the random walk model, the first differences Δ RW are ε_t process.

Fig. 2. Frequency distributions of the largest Lyapunov exponents for Δ STUR and Δ RW processes





Source: Author's calculation.

Table 2. Parameters of the frequency distributions of the largest Lyapunov exponents

	Mean $\bar{\lambda}$	Median	Max	Min	σ	Jarque-Bera (probability)
			m=1			
Δ STUR	0.0017	0.0010	0.0849	-0.0767	0.0264	3.116 (0.211)
Δ RW	-0.0006	-0.0014	0.0864	-0.0973	0.0258	1.436 (0.488)
			m=2			
Δ STUR	0.0024	0.0018	0.0947	-0.0891	0.0273	0.581 (0.748)
Δ RW	-0.0003	0.0009	0.0934	-0.0886	0.0265	1.108 (0.575)
			m=3			
Δ STUR	0.0028	0.0018	0.0860	-0.0690	0.0273	1.464 (0.481)
Δ RW	0.0005	0.0007	0.0997	-0.0953	0.0278	1.273 (0.529)
			m=5			
Δ STUR	0.0021	0.0020	0.0941	-0.0823	0.0270	3.449 (0.178)
Δ RW	0.0008	0.0005	0.0928	-0.0914	0.0264	4.165 (0.125)

			m=7			
Δ STUR	0.0015	0.0018	0.0991	-0.0811	0.0279	0.740 (0.691)
Δ RW	-0.0013	-0.0017	0.0839	-0.1078	0.0274	3.067 (0.216)
			m=10			
Δ STUR	0.0015	0.0019	0.0766	-0.0816	0.0264	0.369 (0.832)
Δ RW	-0.0001	-0.0011	0.0826	-0.0900	0.0258	2.077 (0.354)
			m=15			
Δ STUR	0.0023	0.0025	0.0924	-0.0746	0.0254	0.707 (0.702)
Δ RW	-0.0007	-0.0007	0.0768	-0.0969	0.0249	1.458 (0.482)

Source: Author's calculation.

Next, the obtained results were used to verify a significance of the largest Lyapunov exponent. For Δ STUR process the null hypothesis $H_0 : E(\lambda) = 0$ against the alternative $H_1 : E(\lambda) > 0$ was verified. Since the Jarque-Bera test did not reject the hypothesis of the normal distribution, to verify the significance of exponents the statistic $Z = \frac{\bar{\lambda}}{\sigma}\sqrt{1000}$ was used. The computed values of the *Z*-statistic for different *m* are presented in Table 3. The symbols * and ** indicate values leading to rejection of the null, for the significance levels equal, respectively, 0.05 and 0.01. It is seen that the obtained results indicate the sensitivity to initial conditions. However, it should be marked that the estimated exponents, although positive, are very small indeed (see $\bar{\lambda}$ in Table 2), therefore this identified sensitivity is quite weak.

Different, but expected, results were obtained for Δ RW process. In this case the calculated values of the *Z*-statistic did not lead to rejection of the null hypothesis[7].

Table 3. Values of the *Z*-statistic in the test of a significance of Lyapunov exponents

	m=1	*m*=2	*m*=3	*m*=5	*m*=7	*m*=10	*m*=15
Δ STUR	2.024*	2.832**	3.235**	2.401**	1.655*	1.794*	2.816**
Δ RW	-0.731	-0.327	0.593	0.952	-1.544	-0.182	-0.892

Source: Author's calculation.

The results of this research indicate that the estimation of the largest Lyapunov exponent may be helpful in distinguishing STUR and random walk processes.

Additionally, similarly to R/S analysis, the Kolmogorov-Smirnov test was applied to verify the consistency of the distributions of the largest Lyapunov exponents for Δ STUR and Δ RW processes. The calculated values of *KS*-statistic are presented in Table 4.

[7] For $m = 3$ and $m = 5$ the alternative hypothesis H_1: $E(\lambda) > 0$ was considered whereas in other cases H_1: $E(\lambda) < 0$.

Table 4. Values of the *KS*-statistic in the test of the distributions' consistency of the largest Lyapunov exponents

	m=1	m=2	m=3	m=5	m=7	m=10	m=15
KS	1.185	1.096	0.850	0.738	1.476*	1.163	1.431*

Source: Author's calculation.

The null hypothesis of the consistency of the distributions was rejected for $m = 7$ and $m = 15$ at the significance level $\alpha = 0.05$. This result confirms that the estimation of the largest Lyapunov exponent may be a useful tool of distinguishing STUR and random walk processes, when the appropriate values of the parameter *m* are set. Although the issue of determining such values seems interesting, it is beyond the scope of this paper.

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