

Mateusz Pipień
Cracow University of Economics

The Predictive Value at Risk and Capital Requirements for Market Risk. The Case of PLN/USD Exchange Rate¹

1. Introduction

The main goal of this paper is an application of the Bayesian inference in Value at Risk (*VaR*) prediction. As a *VaR* forecasts we use quantiles of the predictive densities obtained from AR(1)-GARCH(1,1) models (Osiewalski and Pipień 1999, 2003), with α -Stable or Skewed-*t* conditional distributions; Pipień (2005). From the definition, the predictive distribution yields probabilistic and easy to interpret information about ex ante uncertainty of forecasted variables. The predictive density combines the sampling assumptions of considered model, with uncertainty about model parameters. As a result, Value at Risk taken from predictive distributions should yield a flexible tool of risk measuring.

Based on the time series of the PLN/USD exchange rate, we present the dynamics of the predictive Value at Risk estimates obtained in each model. Starting from the dataset consisting of 100 observations, every time when we updated daily observation into dataset, we recalculated posterior distribution of parameters and derive predictive distributions of daily returns of PLN/USD exchange rate. Using various testing procedures (see Kupiec 1995 and Lopez 1999) we compare the accuracy of the predictive VaR estimates among models.

We also present the Bayesian prediction of the minimal capital requirements for market risk. The current regulatory framework, which is deeply based the New Capital Accord, proposed by the Basle Committee on Banking Supervision, allow that financial institutions can use their internal models of *VaR* forecasts as the basis for calculation of the capital requirements. Based on the

¹ Research supported by a grant from Cracow University of Economics.

predictive *VaR* forecasts, obtained in models M_1 and M_2 , we assessed the capital charge for market risk. We check if the quality of the forecasts is sensitive to the type of the conditional distribution in GARCH framework.

2. Predictive Value at Risk Concept

We denote by $y^{(t)}=(y_1, \dots, y_t)$ the vector of observed up to day t (used in estimation in day t) daily growth rates and by $y_f^{(t)}=(y_{t+1}, \dots, y_{t+n})$ the vector of forecasted observables at time t . Let assume, that the following density defines the i -th sampling model ($i=1,2$) at time t :

$$p(y^{(t)}, y_f^{(t)} | M_i, \theta, \eta_i) = \prod_{j=1}^{t+n} p(y_j | M_i, \psi_{j-1}, \theta, \eta_i).$$

Given the prior distribution, i.e. the marginal distribution of all model parameters $p(\theta, \eta_i | M_i)$, the following joint density represents i -th Bayesian model (M_i):

$$p(y^{(t)}, y_f^{(t)} | M_i, \theta, \eta_i) \cdot p(\theta, \eta_i | M_i).$$

From the joint density of parameters and observables it is possible to calculate the posterior distribution of any function of parameters and future daily returns. In particular, we obtain predictive distribution of y_{t+n} , i.e. the distribution of daily return at day $t+n$ conditional given the vector of observed daily returns $y^{(t)}$ and model M_i . We denote the density of the underlying predictive distribution by $p(y_{t+n} | y^{(t)}, M_i)$. It is the marginal density, obtained from the joint predictive distribution of the vector $y_f^{(t)}=(y_{t+1}, \dots, y_{t+n})$:

$$p(y_f^{(t)} | y^{(t)}, M_i) = \int p(y_f^{(t)} | M_i, y^{(t)}, \theta, \eta_i) p(\theta, \eta_i | y^{(t)}, M_i) d\theta d\eta_i. \quad (1)$$

Given predictive density $p(y_{t+n} | y^{(t)}, M_i)$ we define by $VaR_t(\alpha, n | M_i)$ the minus quantile of order α of this distribution:

$$\int_{-\infty}^{-VaR_t(\alpha, n | M_i)} p(y_{t+n} | y^{(t)}, M_i) = \alpha \quad (2)$$

and call it the predictive Value at Risk at time t (calculated on the basis of the vector of observations $y^{(t)}$) with the confidence level α and forecasting horizon n . The quantity $VaR_t(\alpha, n | M_i)$ defines the maximal potential loss of value of the instrument with quoted price x_j , which may occur after n days with predictive probability (in model M_i) equal to α . The properties of the predictive *VaR* definition is studied in Pipień (2005).

According to the Basle Committee suggestions, the general market risk capital requirement C_t is based on Value at Risk estimates calibrated to a ten day forecasting horizon and to probability $\alpha=0.01$. At time t , the capital charge for market risk is equal to maximum of the value of the average Value-at-Risk estimated over the previous sixty trading days (which is approximately one quarter to the trading year), or to the Value at Risk calculated at time t . This

charge is multiplied by a “scaling factor” A_t between three and four. The following formula incorporates predictive Value at Risk into capital charge scheme:

$$C_t = A_t \max \left\{ \frac{1}{60} \sum_{i=1}^{60} \text{Var}_{t-i}(0.01, 10 | M_i), \text{Var}_t(0.01, 10 | M_t) \right\}, \quad (3)$$

where A_t depends on the number of Value-at-Risk exceptions in previous 250 days according to the agreements presented in details by Hendricks and Hirtle (1997).

3. Evaluation of VaR Forecasts

In empirical part of this paper, in order to evaluate predictive VaR forecasts generated from models M_i , we use in particular Kupiec test. This very easy to implement tool of an ex post analysis of Value at Risk forecasts bases on the following likelihood ratio test:

$$LR = 2 \ln[(1 - \hat{\alpha})^{T'+1-S} \hat{\alpha}^S] - 2 \ln[(1 - \alpha)^{T'+1-S} \alpha^S], \quad S = \sum_{t=T}^{T+T'} \xi_t, \quad (4)$$

where $\hat{\alpha}$ is the estimator of the probability of success,

$$\hat{\alpha} = \frac{S}{T'+1} = \frac{1}{T'+1} \sum_{t=T}^{T+T'} \xi_t, \quad (5)$$

of the Bernoulli random variable ξ_t defined in the following way:

$$\xi_t = \begin{cases} 0 & \text{if } y_{t+n} \geq \text{VaR}_t(\alpha, n | M_i) \text{ with prob. } 1 - \bar{\alpha} \\ 1 & \text{if } y_{t+n} < \text{VaR}_t(\alpha, n | M_i) \text{ with prob. } \bar{\alpha}. \end{cases}$$

According to Kupiec criterion (4), given the significance level $x\%$, the model M_i generates acceptably accurate forecasts if $H_0 (\alpha = \bar{\alpha})$ cannot be rejected, otherwise M_i is called inaccurate; see Kupiec (1995), Lopez (1999).

Quite similar approach of measurement of accuracy of the VaR forecasts can be derived from Central Limit Theorem (CLT) for time series $\{\xi_t; t=T, T+1, \dots, T+T'\}$. From CLT we obtain, that:

$$\sqrt{T'+1} \left(\sum_{t=T}^{T+T'} (\xi_t - \bar{\alpha}) \right) \xrightarrow{T' \rightarrow +\infty} N(0, \bar{\alpha}(1 - \bar{\alpha})). \quad (6)$$

Hence, for large values of T' , $\hat{\alpha}$ – defined by (5) – is unbiased and asymptotically normal estimator of $\bar{\alpha}$. The small sample approximation of the variance of the asymptotically distribution of this estimator is given by the formula::

$$d^2(\hat{\alpha}) = \frac{(\bar{\alpha}(1 - \bar{\alpha}))^2}{T'+1}. \quad (7)$$

This leads to the formula fo the standard error of the estimation of $\bar{\alpha}$:

$$d(\hat{\alpha}) = \frac{\hat{\alpha}(1-\hat{\alpha})}{\sqrt{T'+1}}. \quad (8)$$

From asymptotic normality (6), it is possible to build the confidence interval for $\bar{\alpha}$ (given a confidence level $x\%$):

$$\left(\hat{\alpha} - q_x d(\hat{\alpha}); \hat{\alpha} + q_x d(\hat{\alpha})\right), \quad (9)$$

where q_x is the quantile of order $1-x\%$ of the standardised normal distribution. Once again, the model M_i generates acceptably accurate forecasts if – for a given $x\%$ – probability value α does not lie inside the confidence interval, otherwise M_i can be called inaccurate.

As an alternative to the hypothesis-testing framework, Lopez (1999) proposed an evaluation method that uses loss function approach. This approach deeply involves standard forecast evaluation techniques, which are based on how analysed forecasts minimise defined loss function, that represents the evaluator's concern. We considered three different loss functions. As a first we chose very simple binary loss function proposed by Lopez (1999):

$$f_t^{(i)} = \begin{cases} 0 & \text{if } y_{t+n} \geq VaR_t(\alpha, n | M_i) \\ 1 & \text{if } y_{t+n} < VaR_t(\alpha, n | M_i). \end{cases} \quad (10)$$

Additionally, we consider the following loss function:

$$f_t^{(i)} = \begin{cases} 0 & \text{if } y_{t+n} \geq VaR_t(\alpha, n | M_i) \\ 1 + (y_{t+n} - VaR_t(\alpha, n | M_i))^2 & \text{if } y_{t+n} < VaR_t(\alpha, n | M_i). \end{cases} \quad (11)$$

Just like in (10), a score of 1 is imposed in (11) when an exception occurs, but now an additional term, based on the magnitude of the exception, is included. The numerical score increases with the magnitude of the exception and can provide additional information in evaluation of models with comparable number of exceptions. As a result, (11) penalises models more severely as compared to *BL*. Sarma, Thomas and Shah (2003) explain, that (11) is able to express the regulatory concerns in model evaluation. Hence we call (11) the Regulatory Loss (*RL*). However, no score is attached in case if exception does not occur.

The basic application of Value at Risk forecasts is the assessment of risk exposure of financial institution (firm) and also it is a basic point in internal risk management. Since in a firm there is a conflict between the goal of safety (protection from risk) and the goal of profit maximisation, it is of particular interest to build the loss function, which would be able to evaluate *VaR* forecasts from the point of view of these conditions. As suggest Sarma, Thomas and Shah (2003) a mechanism of Value at Risk forecasting, which reported too high values of $VaR_t(\alpha, n | M_i)$, would force the firm to hold too much capital (as a charge for market risk) imposing the opportunity cost of capital upon the firm. They propose the Firm's Loss (*FL*) which penalise *VaR* exceptions, but which also imposes penalty reflecting the cost of capital suffered because of too conservative *VaR* forecasts. The *FL* loss function takes the form:

$$f_t^{(i)} = \begin{cases} c \cdot VaR_t(\alpha, n | M_i) & \text{if } y_{t+n} \geq VaR_t(\alpha, n | M_i) \\ 1 + (y_{t+n} - VaR_t(\alpha, n | M_i))^2 & \text{if } y_{t+n} < VaR_t(\alpha, n | M_i), \end{cases} \quad (12)$$

where $c > 0$ measures the opportunity costs of capital; see Sarma Thomas and Shah (2003). As seen from (12), the numerical penalty, connected with too conservative Value at Risk forecasts, is proportional to $VaR_t(\alpha, n | M_i)$. The coefficient $c > 0$ is time invariant, reflecting assumption of constancy (over time) of the opportunity costs of capital.

In all presented situations the total loss $f^{(i)}$, connected with Value at Risk forecasts in M_i , is equal to sum of $f_t^{(i)}$:

$$f^{(i)} = \sum_{t=T}^{T+T'} f_t^{(i)} \quad (13)$$

4. Competing Volatility Models

In this paper the predictive Value at Risk estimates are obtained within two GARCH models, first with conditional Skewed Student- t (model M_1) and second with α -Stable conditional distribution (model M_2). Both specifications were presented in details by Osiewalski and Pipień (2003) and also by Pipień (2005) respectively. Hence we consider $VaR_t(\alpha, n | M_1)$ and $VaR_t(\alpha, n | M_2)$, where:

$$\int_{-\infty}^{-VaR_t(\alpha, n | M_1)} p(y_{t+n} | y^{(t)}, M_1) = \alpha, \text{ and } \int_{-\infty}^{-VaR_t(\alpha, n | M_2)} p(y_{t+n} | y^{(t)}, M_2) = \alpha.$$

We also analyse the results of the predictive Value at Risk obtained on the basis of much simpler model assumptions, than those forming M_1 and M_2 . As a starting point we assumed, that at time $t+n$ the conditional (given the whole past) distribution of daily returns is a normal distribution with constant variance σ . In model M_3 , we use for Bayesian estimation the whole dataset available at time t (namely $y^{(t)}$), while in case of model M_4 we use only $y^{(t,k)} = (y_{t-k-1}, \dots, y_t)$, for a given k . The resulting forecasts of VaR , namely $VaR_t(\alpha, n | M_3)$ and $VaR_t(\alpha, n | M_4)$ such, that:

$$\int_{-\infty}^{-VaR_t(\alpha, n | M_3)} p(y_{t+n} | y^{(t)}, M_3) = \alpha, \text{ and } \int_{-\infty}^{-VaR_t(\alpha, n | M_4)} p(y_{t+n} | y^{(t,k)}, M_4) = \alpha,$$

can be interpreted as a Bayesian interpretation of methods of Value at Risk forecasts commonly used in risk management practice. Those methods are discussed in Jorion (1996), Best (2000), and applied for Polish financial data by Jajuga, Kuziak, Papla and Rokita (2001).

5. Empirical Results

In this part we present an empirical example of dynamic forecasting of Value at Risk within the Bayesian framework. We considered $T+T'+1=1657$ observations of daily returns on the PLN/USD exchange rates from 05.02.96 till 04.09.02. Starting at $t=T=100$ we calculated posterior distributions of parameters in all competing models, based on dataset $y^{(t)}$ for each $t=T=100$ up to $t=T+T'=1657$. As a result of daily updating observations into $y^{(t)}$ we obtained $T'+1=1558$ posterior distributions of parameters in M_i ($i=1,2,3,4$) and $T'+1=1558$ predictive distributions $p(y_{t+n}|y^{(t)}, M_i)$ ($t=T=100, \dots, T+T'$) for $n=1$ and $n=10$. The main purpose of the following presentation is to check sensitivity of the predictive Value at Risk and capital requirements for market risk with respect to new observations dynamically updated into dataset $y^{(t)}$.

In Table 1 we put the results of evaluation of the Value at Risk forecasts with respect to criteria discussed in Section 3. In each model, based on time series $\{VaR_t(\alpha, n|M_i); t=100, \dots, 1657\}$ for $\alpha=0.01, 0.05$ and 0.1 , we estimated probability of success of the random variable ξ_t (see (12)). We calculated $\hat{\alpha}$ (see (5)) and $d(\hat{\alpha})$ (see (7)) and also p -values of Kupiec (1995) test (p -value LR ; see (4)). The last three columns of Table 1 present evaluation of VaR forecasts based on the loss function analysis. We derived numerical total loss (13) obtained by application of the Binary Loss function (BL ; see (10)), Regulatory Loss (RL ; see (11)) and Firm's Loss (FL ; see (12)).

The presented results clearly shows, that, from the point of view of Kupiec (1995) test, all considered specifications are rejected. Very low values of p -value in all models clearly make each model inaccurate. As an exception we obtained relative great p -value in case of model M_4 , for $k=30$ and $\alpha=0.01$, and model M_3 ($\alpha=0.05$) as well as in models M_4 , $k=5$ and M_3 ($\alpha=0.1$). The results of Kupiec (1995) test, based on the relative long time series $\{VaR_t(\alpha, n|M_i); t=100, \dots, 1657\}$, strongly confirm, that there is no linkage between frequency of days, when daily return exceeds Value at Risk forecast (ex post) and the level of risk defined by probability α .

The evaluation of VaR forecasts done on the basis of the loss function approach yields very distinct results than those generated by testing framework. Evidently, model M_2 (GARCH process with α -Stable conditional distribution) is a source of very conservative forecasts $VaR_t(\alpha, n|M_2)$. In particular, in the whole forecasting period, there were only $BL=3$ exceptions for $\alpha=0.01$ in model M_2 . Other specifications generate relatively greater values of loss BL . With respect to the BL , model M_1 (GARCH process with Skewed Student- t conditional distribution) is the worst (the most liberal) specification in each case of the probability α .

As regards with Regulatory Loss (RL) the ranking of competing specifications is rather unstable with respect to the changes of the value of α . There is no

doubt, that for each values of probability α , Stable-GARCH generates the smallest loss RL . But other models, especially conditionally Skewed Student- t GARCH, change their position in ranking quite dynamically as the level of the probability α changes.

The Firm's Loss function (FL), which captures the opportunity costs of capital, generates very different ranking of specifications, than those obtained on the basis of BL and RL analysis. This loss function appreciates liberal character of VaR forecasts obtained in model M_1 . In turns out, that among all considered specifications, conditionally Skewed Student- t GARCH yields the smallest value of the opportunity cost. Also, very conservative Stable-GARCH is evaluated as the worst source of VaR forecasts.

In each model, M_1, M_2, M_3, M_4 , for $k=5$ and $k=30$, we calculated $VaR_t(0.01, 10|M_t)$ for $t=350, \dots, 1657$. On the basis of the resulting time series of 10 day predictive VaR estimates we assessed the capital charge C_t , according to the formula (3). In Tables 2 and 3 we put in the first column the plots of the capital requirements C_t ($t=350, \dots, 1657$) and, in the second column, the plots of scaling factor A_t ($t=350, \dots, 1657$) obtained in all competing specifications. In order to check the sensitivity of C_t and A_t with respect to the new observations dynamically updated into dataset $y^{(t)}$ in both tables the last row contains daily returns y_t , for $t=350, \dots, 1657$.

As seen from both Tables the capital charge C_t is very sensitive in all models with respect to the new observations updated in $y^{(t)}$. Some occasional outliers and intensification of the volatility make the capital charge to increase rapidly (see Table 1, Figures A and C for t about 350, 650 and 1370), while low volatility period require small amount of capital as a charge for market risk (see Table 1, Figures A and C, for $t=450, \dots, 500, t=800, \dots, 900$). The correlation coefficient between time series C_t ; $t=350, \dots, 1657$ generated by M_1 and M_2 equals 0.65). In the regions of unexpected jumps of PLN/USD exchange rate (namely for t about 650 and 1370), in spite of significant differences in forecasting mechanisms discussed above, both models predict quite similar minimal capital charge C_t . For $t=645$ the capital charge obtained from M_1 , as well as from M_2 , crosses 7% and for $t=1370$ it is very close to 7% in both models. Apart from these situations M_2 generates higher average level of the capital charge than model M_1 . The mean C_t equals 4.58% in model M_2 while in M_1 it does not cross the value 3.78%.

The fundamental differences between models M_1 and M_2 in the capital charge forecasts are shown on Figures B and D in Table 2, presenting dynamic behaviour of scaling factor A_t . Since M_2 generates very conservative $VaR_t(0.01, 10|M_t)$ forecasts, it remains in the safe green zone for the whole period $t=350, \dots, 1657$. Hence, the scaling factor is constant and equals $A_t=3$ for each t . In case of model M_1 we observe volatile behaviour of A_t as the new observations are updated into dataset. Initially, for $t=320, 820$, the number of exceptions K is very high. Hence $A_t=4$, for $t=320, 820$, and M_1 reaches the risky red zone. We observe, that for $t=800, \dots, 1100$, the number of exceptions K sig-

nificantly decreases, making model M_1 safer. For t greater than 1100 Skewed- t GARCH model does not reach the red zone, remaining in the yellow zone or, for quite short period of time, in the green zone (for t less than 1200).

Table 3 presents time series of C_t and A_t obtained in models assuming normality of sampling predictive density of y_{t+n} . The dynamics of capital charge clearly depends on the number of observation included in likelihood in models M_3 and M_4 . In case of model M_4 , which uses only $k=5$ past observations in estimation at time t , the resulting time series of C_t is very sensitive with respect to the new observations dynamically updated into $y^{(t,k)}$; see Figure C. Figure E in Table 3 shows, that model M_3 , which uses in estimation the whole series $y^{(t)}$, generates capital charge estimates rather insensitive with respect to the new observations. After including more than $t=1100$ observed daily returns, the total impact of new, and even very volatile, observations seems to be very weak.

Also the dynamics of scaling factor, reflecting the safety of each model's VaR forecasts, clearly depend on the number of observations used in estimation. We observe, that model M_4 , for $k=30$ does not reach the red zone, remaining quite safe source of predictive Value at Risk estimates; see Figure B. The average scaling factor in case of models M_3 and M_4 , for $k=5$ is relatively greater, than this obtained in M_4 , for $k=30$. As a result, M_3 and M_4 , for $k=5$ are subject to the higher average capital charge than in case of M_4 , for $k=30$.

References

- Best P. (2000) *Wartość narażona na ryzyko*, Oficyna Ekonomiczna ABC, Kraków.
- Hendricks D., Hirtle B. (1997) Bank Capital Requirements for Market Risk: The Internal Model Approach, *Federal Reserve Bank of New York Review*.
- Jajuga K., Kuziak K., Papla D., Rokita P. (2001) Ryzyko wybranych instrumentów polskiego rynku finansowego, *Rynek Terminowy* 11, 133-140.
- Jorion P. (1996) *Value-at-Risk*, McGraw Hill, New York.
- Kupiec P. (1995) Techniques for Verifying the Accuracy of Risk Management Models, *Journal of Derivatives* 2, 173-184.
- Lopez J.A. (1999) Methods for Evaluating Value-at-Risk Estimates, *Federal Reserve Bank of San Francisco Economic Review*, 3-17.
- Osiewalski J., Pipień M. (1999) Bayesian Forecasting of Foreign Exchange Rates Using GARCH Models with Skewed- t Conditional Distributions, *25-th International Conference MACROMODELS'98*, (ed.) Władysław Welfe, Łódź, 195-218.
- Osiewalski J., Pipień M. (2003) Univariate GARCH processes with asymmetries and GARCH-In-Mean effects: Bayesian analysis and direct option pricing, *Przegląd Statystyczny* 50, 5-29
- Pipień (2005) Value at Risk Estimates and Capital Requirements for Market Risk Obtained From GARCH Predictive Densities, *Acta Universitatis Lodzianis Folia Oeconomica* 190, Łódź, 213-227.
- Sarma M., Thomas S., Shah A. (2003) Selection of VaR Models, *Journal of Forecasting* 22, 337-358.

Table 1. Evaluation of VaR forecasts $VaR_t(\alpha, 1|y^{(t)}, M_t)$ for $\alpha=0.01, 0.05$ and 0.1 .

	$\hat{\alpha}$	$d(\hat{\alpha})$	p-value I	BL	RL	FL (c=1)
$\alpha=0.01$						
M_1	0.0334	0.0046	1.40e-7	52	266.11	2010.88
M_2	0.0020	0.0011	1.81e-13	3	125.11	3481.88
$M_4; k=30$	0.0173	0.0033	0.0133	27	258.80	2442.11
$M_4; k=5$	0.0308	0.0044	1.00e-6	48	281.76	2339.58
M_3	0.0205	0.0040	0.0017	32	297.48	2120.62
$\alpha=0.05$						
M_1	0.0802	0.0069	5.60e-6	125	408.92	1573.55
M_2	0.0154	0.0031	<1e-18	24	250.36	2244.90
$M_4; k=30$	0.0353	0.0047	8.34e-4	55	294.70	1830.55
$M_4; k=5$	0.0629	0.0065	0.0180	98	344.42	1806.10
M_3	0.0539	0.0057	0.2469	84	436.02	1678.79
$\alpha=0.1$						
M_1	0.1444	0.0089	3.06e-7	225	509.71	1333.86
M_2	0.0424	0.0051	<1e-18	66	388.51	1813.80
$M_4; k=30$	0.0738	0.0066	3.86e-5	115	442.08	1602.86
$M_4; k=5$	0.0950	0.0074	0.0250	148	413.28	1549.46
M_3	0.9031	0.0074	0.1732	145	504.22	1430.73

Table 2. Capital requirements for exchange rate risk C_t ($t=100, \dots, 1657$) and scaling factor A_t obtained in M_1 and M_2 .

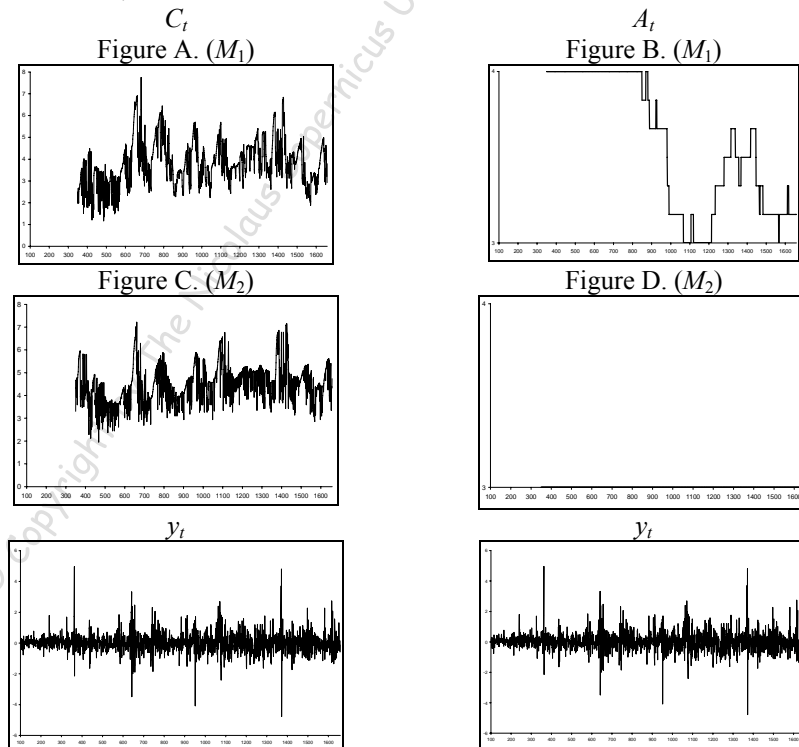


Table 3. Capital requirements for exchange rate risk C_t ($t=100, \dots, 1657$) and scaling factor A_t obtained in M_3 and M_4 (for $k=30$ and $k=5$).

