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Estimating the Volatility of the Stock Index WIG20 with Weak-GARCH and Diffusion GARCH Models

1. Introduction

Financial instrument volatility is nowadays the central object of financial econometrics research and the key variable in financial mathematics models. However, there exists some discrepancy between the methodologies applied to describe the volatility dynamics in these two fields contributing to modern finance. Financial mathematics models are usually defined in continuous time, while financial data analyzed in financial econometrics (stock price returns, indices, exchange rates) are typically observed as discrete processes. In consequence, the results obtained by financial mathematicians and financial econometricians come from the different worlds, and very often they are expressed in completely different languages. The question how to translate one to another is very important for finance theory and applications.

The volatility of a financial instrument is usually defined as the conditional variance of the return on it, upon the information available on period before. The family of GARCH models is the most popular tool used by financial econometricians for modeling volatility. Although GARCH models have many pleasant properties, their disadvantage is that they do not form the class of models closed under temporal aggregation. This is very inconvenient, especially in the situation when one wants to establish the correspondence between GARCH and some continuous time models. The results of Nelson (1990), Drost and Nijman (1993), and Drost and Werker (1996) show that there exists the class of models encompassing the family of GARCH models and satisfying the requirements of temporal aggregation and correspondence to some continuous time models. Drost and Nijman (1993) called these models weak GARCH.

The family of weak GARCH models is very useful. As a result of the properties mentioned above, it builds a bridge between discrete time GARCH models and continuous time diffusion processes (Drost and Werker 1996). This gives a methodological support to the theory of realized volatility (Andersen and Bollerslev 1998). The importance of weak GARCH models is also connected with the fact that they can form a frame for some models describing return series sampled at unequal time intervals, for instance ACD-GARCH models (Ghysels and Jasiak 1998).

In the paper we investigate the dynamics of the stock index WIG20 volatility. We consider return series calculated in different frequencies. For each return series we estimate GARCH(1,1) model and, basing on its parameters, we calculate the corresponding diffusion GARCH model parameters. The question is whether the dynamics of the WIG20 is driven by some kind of diffusion GARCH process. We try to check it by calculating the diffusion GARCH parameters based on the GARCH(1,1) model fitted to the returns in different frequencies. Our results do not reject that presumption.

2. Weak GARCH Models

We consider logarithmic percentage returns

$$R_t = 100(\ln P_t - \ln P_{t-1}),$$

where P_t is the closing quotation of a financial instrument on period *t*.

The volatility of financial instrument on period *t* is defined as the conditional variance of the return, upon the set of information Ω_{t-1} available on period *t*-1, i.e.

$$\sigma_t^2 = E((R_t - E(R_t \mid \Omega_{t-1}))^2 \mid \Omega_{t-1}).$$

The most frequently used volatility model is the GARCH(*p*, *q*) model (Bollerslev 1986) with the specification given by the formulas

$$y_t = \sigma_t \varepsilon_t, \tag{1}$$

$$\sigma_t^2 = \psi + \sum_{i=1}^{q} \alpha_i y_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2, \tag{2}$$

where $\varepsilon_t \sim \mathrm{iid}(0,1)$, $\psi > 0$, $\alpha_i \geq 0$ and $\beta_j \geq 0$.

The class of weak GARCH models was first defined in (Drost, Nijman 1993). A sequence of uncorrelated random variables y_t with zero mean is a weak GARCH(*p*, *q*) process if the σ_t^2 defined by equation (2) is an orthogonal projection of the variable y_t^2 on the closed linear subspace spanned by $\{1, y_{t-1}, y_{t-2}, \dots y_{t-1}^2, y_{t-2}^2, \dots\}$. In the foregoing definition we consider the equiva-

lence classes of random variables created by the relation of equality almost sure as elements of a Hilbert space with scalar product given by the expected value of the usual variable product (Brockwell, Davis 1995).
According to the definition, we have

$$E\left(y_t^2-\sigma_t^2\right)=E\left[\left(y_t^2-\sigma_t^2\right)y_{t-i}\right]=E\left[\left(y_t^2-\sigma_t^2\right)y_{t-i}^2\right]=0\text{, for } i=1,2,\ldots.$$

The relaxation of conditions required in the definition of GARCH model gives as a result a family of volatility models closed under temporal aggregation. It is obvious from the definitions that GARCH models form a subset of the weak GARCH family.

The most popular GARCH model is GARCH(1,1). Drost and Nijman (1993) showed that the class of symmetric weak GARCH(1,1) processes is closed under temporal aggregation. If the process y_t follows the symmetric week GARCH given by

$$\sigma_t^2=\psi+\alpha y_t^2+\beta\sigma_t^2, \qquad t=0,1,2,\ldots \tag{3}$$

then the parameters $\alpha_{(h)}$ and $\beta_{(h)}$ in the weak GARCH(1, 1) equation

$$\sigma_{(h)th}^2=\psi_{(h)}+\alpha_{(h)}y_{th-h}^2+\beta_{(h)}\sigma_{(h)th-h}^2, \quad t=0,1,2,\ldots \tag{4}$$

for the process y_{th} depend only on the values α and β. The plots in figure 1-2 present the values of parameters $\alpha_{(h)}$ and $\beta_{(h)}$ for different h and fixed α and β.

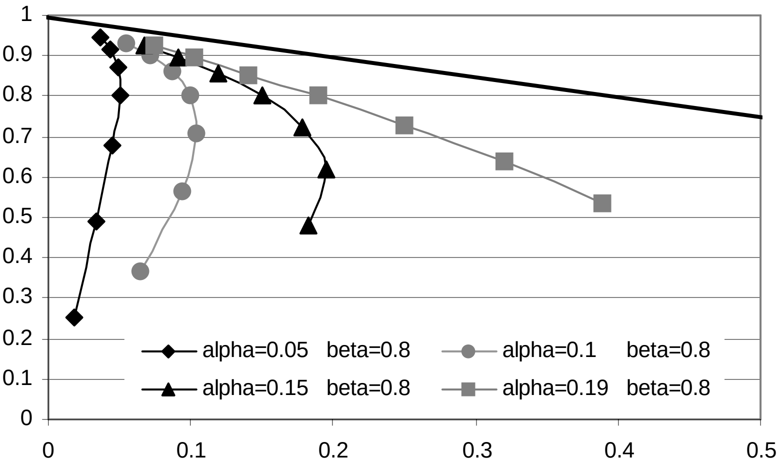

Fig. 1. Aggregation effect on weak GARCH(1,1) parameters. Dependence of $\alpha_{(h)}$ (horizontal axis) and $\beta_{(h)}$ on frequency $h=8,4,2,1,\frac{1}{2},\frac{1}{4},\frac{1}{8}$ (the direction of points is from the bottom to the top) and the values of $\alpha=\alpha_{(1)}$ and $\beta=\beta_{(1)}=0.8$

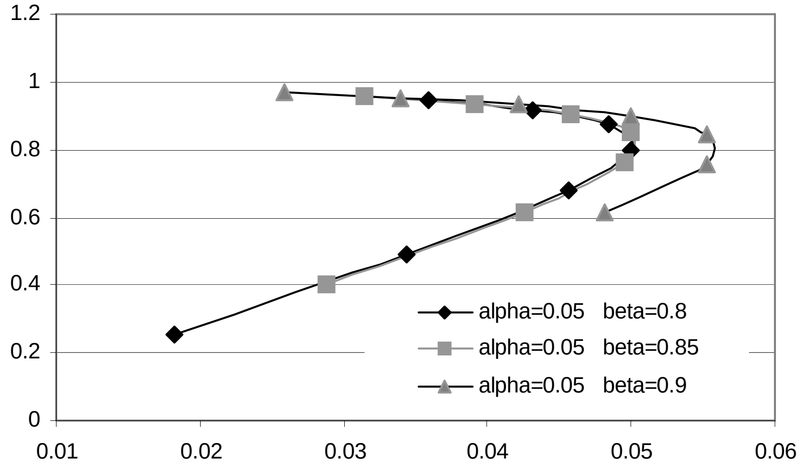

Fig. 2. Aggregation effect on weak GARCH(1,1) parameters. Dependence of $\alpha_{(h)}$ (horizontal axis) and $\beta_{(h)}$ on frequency $h = 8, 4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ (the direction of marked points is from the bottom to the top) and the values of $\alpha = \alpha_{(1)} = 0.05$ and $\beta = \beta_{(h)}$

3. The Correspondence Between Weak GARCH and Diffusion GARCH Models

A continuous stochastic process $Y(t)$ ($t \geq 0$) is called a diffusion GARCH process if for every initial point t_0 and $h > 0$, the discrete process $Y(t_0 + t + h) - Y(t_0 + t)$ ($t = h, 2h, 3h, \ldots$) is the weak GARCH process with parameter vector $\zeta_h = (\psi_h, \alpha_h, \beta_h, \kappa_h)$. In addition, it is assumed that the ζ_h is a continuous function of the argument *h* and that the function

$$f(h) = \frac{E\left((Y(t+h) - Y(t))^4\right)}{h^2} \tag{5}$$

is bounded.

In this paper we consider the simple diffusion process given by the following stochastic equation system

$$dY(t) = \sigma(t)dW_1(t), \tag{6}$$

$$d\sigma^2(t) = \theta\left(\omega - \sigma^2(t)\right)dt + \sqrt{2\lambda\theta}\,\sigma^2(t)dW_2(t). \tag{7}$$

Here W_1, W_2 are two independent Brownian motions, and the parameters satisfy the restrictions $\omega > 0$, $\theta > 0$, $\lambda \in (0,1)$.

Nelson (1990) has shown that the process defined by (6)-(7) can be approximated by a sequence of discrete GARCH processes with innovations iid $N(0,1)$. The approximating process with index h is defined in the points $h, 2h, \ldots$ Drost and Werker (1996) proved that any aggregation of such processes is a weak GARCH process. Moreover, every diffusion GARCH given by (6)-(7) with parameter vectors $\zeta_h = (\psi_h, \alpha_h, \beta_h, \kappa_h)$ can be described using only three parameters $\omega > 0$, $\theta > 0$, $\lambda \in (0,1)$ that satisfy the following equations (8)-(11)

$$\psi_h = h\omega(1 - \exp(-h\theta)), \tag{8}$$

$$\alpha_h = \exp(-h\theta) - \beta_h, \tag{9}$$

$$\kappa_h = 3 + 6\frac{\lambda}{1-\lambda}\frac{\exp(-h\theta) - 1 + h\theta}{(h\theta)^2}, \tag{10}$$

$$\frac{\beta_h}{1+\beta_h^2} = \frac{c_h \exp(-h\theta) - 1}{c_h(1 + \exp(-2h\theta)) - 2}, \tag{11}$$

where $c_h = \dfrac{4(\exp(-h\theta) - 1 + h\theta) + 2h\theta(1 + h\theta(1-\lambda)/\lambda)}{1 - \exp(-2h\theta)}$.

Knowing a parameter vector $\zeta_h = (\psi_h, \alpha_h, \beta_h, \kappa_h)$ for some, $h > 0$, one can show that the parameters ω, θ, λ of the corresponding diffusion GARCH model are given by

$$\theta = -\frac{1}{h}\ln(\alpha_h + \beta_h), \tag{12}$$

$$\omega = \frac{1}{h}\psi_h(1 - \alpha_h - \beta_h)^{-1}, \tag{13}$$

and

$$\begin{aligned}\lambda &= 2\alpha_h \ln^2(\alpha_h + \beta_h)[1 - \beta_h(\alpha_h + \beta_h)]\times \\ &\times \left\{\left[1 - (\alpha_h + \beta_h)^2\right](1-\beta_h)^2 + \alpha_h[1 - \beta_h(\alpha_h + \beta_h)]\times\right. \\ &\times \left.\left[6\ln(\alpha_h + \beta_h) + 2\ln^2(\alpha_h + \beta_h) + 4(1 - \alpha_h - \beta_h)\right]\right\}^{-1}.\end{aligned} \tag{14}$$

4. The Data

Our data consist of the quotations of the stock index WIG20 for the period from November 10, 1998 to March 31, 2005 (1601 observations). The time series under scrutiny are the series of percentage logarithmic return series calculated with frequency 1 day, 2, 3, 4 days, and 1 week. Descriptive statistics of the considered returns are presented in table 1.

Table 1. Descriptive statistics of the analyzed return series

Returns	Mean	Maximum	Minimum	Standard deviation	Skewness	Kurtosis
1-day	0.0279	7.0368	-9.7033	1.6761	0.0453	4.9243
2-day	0.0547	9.8815	-9.6712	2.4207	0.0849	4.5546
3-day	0.0811	8.8643	10.8282	2.7823	-0.2771	3.9249
4-day	0.1093	13.2628	-15.3802	3.4648	-0.0331	5.4958
1-weak	0.1367	14.5012	-16.3032	3.8938	-0.1144	5.0562

5. Empirical Results

Table 2 presents the parameters of GARCH(1,1) model fitted to the analyzed return series and parameters of diffusion GARCH models corresponding to them by equations (12)-(14). All calculations were proceeded with packages TSMod 3.23 (Davidson 2003) and SDE Solver (Janicki and Izydorczyk 2001).

Assuming that the quotations of WIG20 can be described by the diffusion model (6)-(7), we could expect that the parameters of those models based on weak GARCH models fitted to the returns in different frequencies would look similar.

Table 2. Parameters of estimated GARCH and diffusion GARCH models

Return	Frequency	GARCH parameters			Diffusion GARCH parameters		
	$1/h$	ψ	α	β	ω	θ	λ
1-day	1	0.0264 (0.0135)	0.04672 (0.0099)	0.9425 (0.0123)	2.5399	0.0105	0.3198
2-day	1/2	0.1056 (0.0593)	0.0576 (0.0118)	0.9220 (0.0125)	2.5807	0.0103	0.2982
3-day	1/3	0.02079 (0.1304)	0.0596 (0.0171)	0.9124 (0.0229)	2.4706	0.0095	0.2662
4-day	1/4	0.4157 (0.3431)	0.0853 (0.0296)	0.8811 (0.0398)	3.0957	0.0085	0.4323
1-weak	1/5	1.9793 (1.5515)	0.1381 (0.0524)	0.7366 (0.1054)	3.1558	0.0268	0.5467
1-weak	1/7	1.9793 (1.5515)	0.1381 (0.0524)	0.7366 (0.1054)	2.2544	0.0191	0.5466

Our empirical results seem to be consistent with the above-mentioned rule when we consider 1-day, 2-day and 3-day returns. For those three return series, the corresponding parameters of diffusion GARCH models differ very little from each other. The 4-day returns and weekly returns do not fit this pattern. Thus, the results of our investigation are rather ambiguous. There remains opened the question what should be the proper value of the frequency $1/h$ corresponding to the weekly return (h=1/5 or h=1/7). This important problem connected with the irregularly spaced data and market microstructure effects probably could be partially solved by using an Autoregressive Conditional Duration (ACD) model (Ghysels and Jasiak 1998).

Table 3 compares the sample kurtosis of analyzed return series with the theoretical value of kurtosis obtained from GARCH diffusion models. Apart from the case of weekly returns, the estimates of kurtosis are very similar to the empirical values.

Table 3. Unconditional kurtosis induced by diffusion model and sample kurtosis

Return	1-day	2-day	3-day	4-day	1-week
Model kurtosis	4.4055	4.2726	4.0872	5.2829	6.6117
Sample kurtosis	4.9243	4.5546	3.9249	5.4958	5.0562

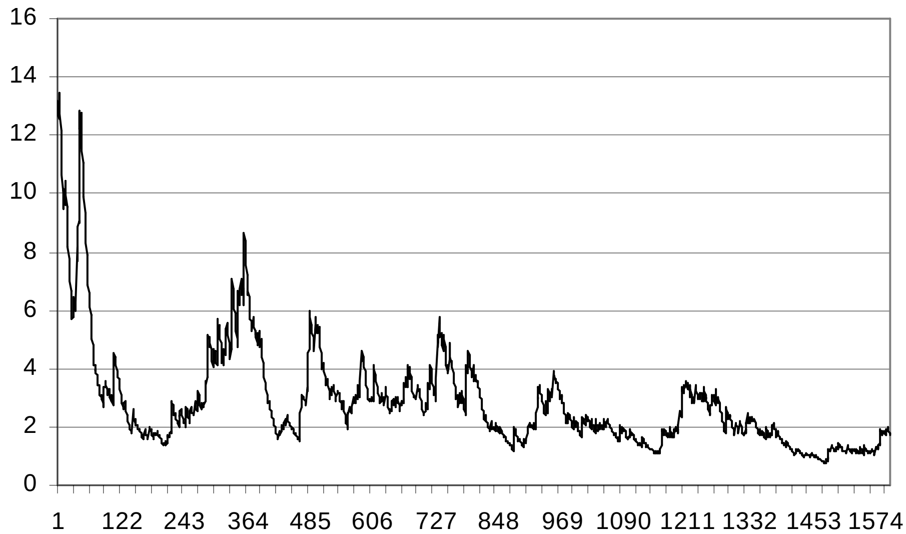

Fig. 3. Daily volatility estimates from GARCH(1,1) model

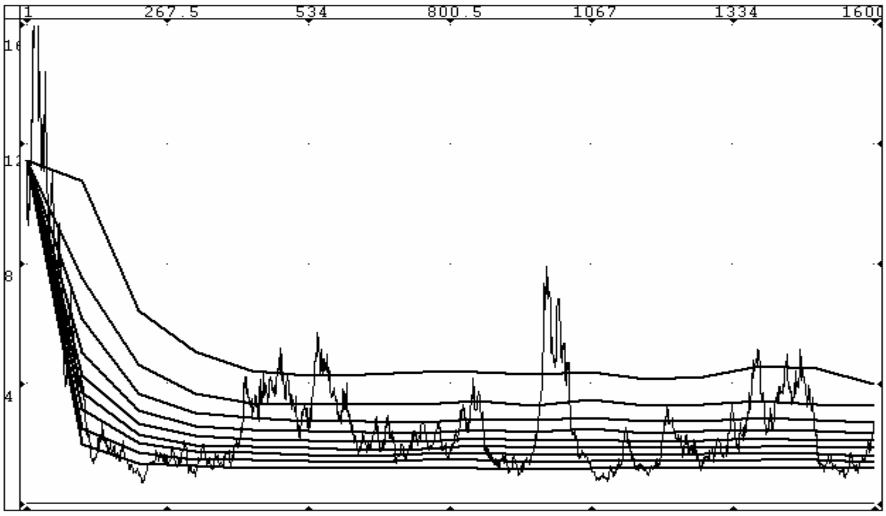

Fig. 4. Simulation of the daily volatility trajectory obtained from diffusion GARCH model for daily returns (with the quantile lines)

6. Conclusions

Financial instrument volatility is the key notion in theoretical finance. There are two different ways to describe it: first one with discrete time volatility models (for example GARCH) and the second one with continuous time financial mathematics models. In applications, it is very important to have a possibility of comparing the results obtained by these two different approaches.

Introduced by Drost and Nijman (1993), the class of weak GARCH models includes GARCH models and satisfies the requirements of temporal aggregation and correspondence to some continuous time models. As a result of these properties, it can be considered as a bridge between discrete time GARCH models and continuous time diffusion processes (Drost and Werker 1996).

In the paper we apply weak GARCH models to describe the volatility dynamics of the stock index WIG20. We consider return series calculated in different frequencies. For each return series we estimate weak GARCH(1,1) model and basing on its parameters we calculate the corresponding diffusion GARCH model parameters. We check if the property of temporal aggregation is in accordance with the models fitted to real financial data. The next question we ask is whether the dynamics of the WIG20 is driven by some kind of diffusion GARCH process. Our results, though slightly ambiguous, do not reject that presumption.

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