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Stability of Equlibrium Point in the Case of Solow's Model

1. Introduction. Problems Connected with Economic Modelling and Stability of Model Solutions

From the beginning of XIXth century economists started using mathematical equations to describe economic phenomena. Examples of such equations may be found in works of Antoine Cournot, Leon Walras, John Maynard Keynes[1] and Alfred Marshall[2]. At the beginning, equations were simple, later they were endowed with stochastic structure and finally took the form of systems of ordinary differential equations[3]. Though, one can say that economic phenomena are too specific to use such equations successfully, many economists are convinced of their usefulness in theory and practice.

It has to be mentioned, that seeked functions have to be continuos functions of some variable, for example the time variable. With the assumption of continuos time we obtain some useful properties of dynamical systems, modelled by the use of continuos and differentiable functions[4]. It is very important to remember about differences between reality and its copy in the form of model, as well as, about forecasting problems for the states of variable process while taking specific unit of time, for example – one year.

In this paper we analyse stability problems for an economic model. It has the form of a system of ordinary differential equations. Assume that its solution

[1] See: Keynes (1956).
[2] See: Marshall (1925).
[3] See: Medio (1993), Gandolfo (1997), Lorenz (1997), Shone (1997), Tokarski (2003), Romer (2000).
[4] See: Krasiński (2001).

and its properties are known. We shall see what will happen, when its solution is unstable.

We now remind what are the disadvantages of the data that are used as a initial conditions in such research. Since it is impossible to eliminate measurement errors, we could be sure, that our initial conditions are not the same as the real ones. By solving system equations and finding a function that satisfies given initial conditions, we make an unestimated error. The distance between that function and the function representing real trajectory of a system may tend to infinity.

Testing stability of solutions of mathematical economy models given by systems of differential equations is necessity. All trying to control economic systems without knowledge about stability of equilibrium points are doomed to failure.

We shall not discuss the problem of interaction between the decisionally used empirical model and economic reality response. If data allow to formulate some statistical hypothesis about the model and testing them, then by using the model for taking decisions we could talk about theory which has an influence on economic reality. In other cases theory remains only theory.

The purpose of this paper is a presentation of results of calculations concerning equilibrium point for Solow's model with more general assumptions that the classic ones and testing its stability for the Polish economy data. In addition, the paper contains discussion about possibilities of such verifications and extending the domain of Solow's model theory.

2. Solow's Model Characteristics

Growth theory model, proposed by Solow[5] in his fundamental article „*A Contribution to the Theory of Economic Growth*" were constructed with some assumptions. Solow wrote: „All theory depends on assumptions which are not quite true. That is what makes it theory[6]". We will make derivation of more general formula. Shortly, we will write:

$$\dot{X}(t) \equiv \frac{dX(t)}{dt} .$$

Solow considered growth of production factor, in the form of growth of „capital". Because of ambiguity of this concept[7] we suppose, that he thought about capital assets. Economic theory specifies capital growth in the form of equation:

[5] See: Solow (1956).
[6] See: Solow (1956).
[7] See: Milo, Bieda, Leszczyk, Miler, Witkowska (2004).

$$\dot{K} = I - \lambda \cdot K \ ,$$

where:
K – amount of capital assets involved in production process,
I – investments in capital assets,
λ – coefficient of depreciation of capital assets, $\lambda \in (0\,;1)$.

We have to add, that Solow in his paper didn't explicitly write about coefficient of depreciation.

Investments I are dependent on a value of output Y and a coefficient s called by Solow a savings rate. They are connected by the relation:

$$I = s \cdot Y \ ,$$

where: s – savings rate[8], $s \in (0\,;1)$, Y – value of output.

The last equation follows from savings and investments equality assumption. Under this assumption s is also investment rate.

Notice, that we make an assumption of persistence of coefficient of depreciation and investment rate. This assumption is too oversimplifying, but we accept it temporarily.

Output is generated by production function F, which is a two factors function in Solow's article. This function is required to be at least C^2 with positive derivative of first order, satisfying Inada's conditions and is homogenous of first order. We accept these assumptions and write:

$$Y = F(K, L) \ ,$$

where: Y – output, K – amount of capital assets, L – amount of labour supply.

Notice, that we skip for a moment some other common assumptions about production function, e. g. negative second partial derivatives, other production factors. Accepted assumptions are sufficient to the following considerations and will give rise to much stronger results[9].

Dividing both sides of last equation by amount of L and using the first order homogenous function F, we have:

$$\frac{Y}{L} = \frac{1}{L} F(K, L) = F(k, 1) \ ,$$

[8] See: Solow (1956), Tokarski (2003).

[9] We need to make a comment about variable L. Saying „labour supply" causes ambiguity. It is hard to identify it with amount of employment or labour force participation rate. There is no reliable and good enough empirical measure of labour. Thus we use in our considerations an abstract production factor, which we call „labour". In the next parts of the text, we will use interchangeably expressions „labour supply" and „amount of employment".

where $k = \frac{K}{L}$ is a capital-labour ratio. Amount of it is a variable, which is taken into further considerations. By definition, k is an amount of capital for one unit of labour supply. Its changes inform us about changes in proportion between this two production factors.

Notice, that:

$$\dot{k} = \frac{d}{dt}\left(\frac{K}{L}\right) = \frac{\dot{K}L - K\dot{L}}{L^2} = \frac{\dot{K}}{L} - \frac{K}{L}\frac{\dot{L}}{L} = \frac{\dot{K}}{L} - k\frac{\dot{L}}{L}.$$

Finally, we have:

$$\dot{k} = \frac{s \cdot F(K,L) - \lambda \cdot K}{L} - k\frac{\dot{L}}{L} = s \cdot F(k,1) - k \cdot \left(\lambda + \frac{\dot{L}}{L}\right).$$

The last equation is a general version of Solow's equation. Production function F and labour supply growth rate is not given, so we cannot solve this equation, unless some assumptions about them will be made. Then we could discuss stability of equlibrium points. It can be proved, that the last equation (under some assumptions about derivative of RHS) could always be locally solved.

Let us find a stationary point, that is a point, which satisfy the condition:

$$\dot{k} = 0.$$

It means, that the stationary point k^* satisfy equation:

$$s \cdot F(k^*,1) - k^*\left(\lambda + \frac{\dot{L}}{L}\right) = 0.$$

Stability analysis for k^* can be done due to the linearization of movement equation in a neighbourhood of the point k^* thanks to Taylor's theorem and checking stability conditions taken from Lapunow's theorem. It is important to remember about some disadvantages of this procedure[10].

Let:

$$H(k) = s \cdot F(k,1) - k\left(\lambda + \frac{\dot{L}}{L}\right).$$

From the above-mentioned theorems it follows:

$$\frac{dH(k)}{dk}(k^*) = s \cdot \frac{dF(k,1)}{dk}(k^*,1) - \left(\lambda + \frac{\dot{L}}{L}\right) = s \cdot \frac{dF(K,L)}{dK}(k^*,1) - \left(\lambda + \frac{\dot{L}}{L}\right).$$

[10] See: Milo, Malaczewski (2004).

That coefficient is a linearized form of our equation. To check stability let us consider:

$$s \cdot \frac{dF(K,L)}{dK}(k^*,1) - (\lambda + \frac{\dot{L}}{L}) < 0 \quad \text{or} \quad \frac{dF(K,L)}{dK}(k^*,1) < \frac{(\lambda + \frac{\dot{L}}{L})}{s}.$$

On the other hand, we have:

$$\frac{dF(K,L)}{dK}(k^*,1) > 0.$$

To analyse the last inequality we have to find the value of production factor L connected with labour and value of its derivative, and, thanks to assumption of persistence coefficient of depreciation of capital assets and investment rate, we may estimate values of the last two variables. Then, the upper bound in the last inequality means, that for its fulfillment it is needed that the stationary value of k^* would not be too close to zero. Otherwise, values of derivative (what is a result of Inada conditions) tend to infinity.

3. Empirical Verification of Solow's Model Equlibrium Point

There are a few questions which need to be answered during process of empirical verifications of Solow's model equlibrium point. First of all, researcher have to decide what variable will be used in an estimation process. This choice problem is connected with ambiguity of some abstract expressions, like „capital" or „labour". In our research – authors took value of capital assets as a „capital", and amount of employment as a „labour". It is obvious that these measures are not perfect.

Another problem, characteristic for Poland is the length of time series. When we are trying to use annual data, we can sensibly use at most 10 observations. It is not always possible for us to change yearly data to quarterly ones. Here we used quaterly data for fixed capital K and employment L quarterly series[11].

From capital growth Solow's equation we can infer, that „investments" means net investments. There are no such time series published for Poland, so we took a sum of investments outlays and continued investments as „investments".

Above-mentioned problems force us to make a question: is it reasonable to verify such hypotheseis for the Polish economy. All methods used here to solve these problems are not perfect and obtained estimation results have to be treated carefully.

[11] See: Milo, Bieda, Leszczyk, Miler, Witkowska (2004).

It is controversial verifying some hypothesis that come from mathematical economy models by using discrete econometric techniques. Some researchers feel doubts about this approach. Notice, that verification of economic hypothesis is the most important task for econometry as a science.

Necessary parametres were estimated for the data taken from the UL - Department of Econometrics database. The sample period covers data from the first quarter of 1994 to the fourth quarter of 2003. In some estimations the sample size changes due to availibility of data. Computations were done by the use of *E – views*®. The following notation will be used further:

$$\Delta X_t = X_t - X_{t-1}.$$

The parameters of our model were estimated by using the following equations:

a) labour supply growth rate equation:

$$\Delta L_t = n \cdot L_t + \xi_{1t},$$

where: L_t – amount of labour supply in a moment t, ξ_{1t} – random term, n – labour supply growth rate;

b) investments rate equation:

$$I_t = s \cdot Y_t + \xi_{2t},$$

where: I_t – sum of investments outlays and continued investments in period t, Y_t – real Polish GDP in period t, ξ_{2t} – random term;

c) coefficient of capital assets depreciation were estimated from the following equation:

$$\Delta K_t = I_t + (-\lambda) \cdot K_t + \xi_{3t},$$

where: K_t – amount of physical capital in period t, I_t – sum of investments outlays and continued investments in period t, ξ_{3t} – random term;

d) sample mean values of derivative of production function F taken with respect to the first production factor were estimated from the following equation:

$$\Delta Y_t = \alpha \cdot \Delta K_t + \xi_{4t},$$

where: K_t – amount of physical capital in period t, Y_t – real GDP in period t, ξ_{4t} – random term.

Computations gave the following results (table 1):

Table 1. Estimates parameters and corresponding *t* - values

Model	Sample size	Parameter	t-value	Significance
$\Delta L_t = n \cdot L_t + \xi_{1t}$	1994:2 2003:4	-0.003268	-1.354930	0.1834
$I_t = s \cdot Y_t + \xi_{2t}$	1994:1 2003:4	0.103379	16.18145	0.0000
$\Delta K_t = I_t + (-\lambda) \cdot K_t + \xi_{3t}$	1994:2 2000:4	-0.014905	-1.758359	0.0905
$\Delta Y_t = \alpha \cdot \Delta K_t + \xi_{4t}$	1994:2 2000:4	0.151716	1.551301	0.1329

Source: author's own calculations.

The estimates have usually low statistical significance. Due to this, examinations of stability were done twice – first with estimated values and another – with zeroes where at significance level 0,1 there was no rejection of null hypothesis.

We are testing the following inequality:

$$\alpha < \frac{\lambda + n}{s}.$$

The first case:

$$0.152 < \frac{0.015 - 0.003}{0.103} = 0.117.$$

The second case:

$$0 < \frac{0.015}{0.103} = 0.146.$$

In the first case the estimated parameters shows us the lack of stability of Solow's model equlibrium point, but in the second case the stability exists. Differences of these two cases are significant.

4. Possibilities of further extensions of Solow's model

The discussed formula of Solow's model equation is not in its most general form. In the equation the investment rate and coefficient of depreciation of capital assets are treated as constant in a whole assumed and considered time, and even constant until infinity. This assumption is too strong for us. Treating these parametres as continuos functions of time will not change analysis significantly. However, if we treat them as functions of capital-labour ratio the whole analysis would be different. There are some economic circumstances validating such analysis. Mathematically it would mean, that some ingredients will appear during calculations of right-hand-side derivative, i. e.

$$\frac{dH(k)}{dk} = s(k) \cdot \frac{dF(k,1)}{dk} + F(k,1) \cdot \frac{ds(k)}{dk} - (\lambda(k) + \frac{\dot{L}}{L}) - k \cdot \frac{d\lambda(k)}{dk}.$$

It has to be considered whether these new ingredients will have significant influence for the sign of the whole right-hand-side. The answer for this problem needs further studies.

Another important issue is the degree of influence of assuming constancy of „labour" production factor growth rate. Solow assumed it is a constant. It leads to an exponential labour function. And again in our opinion this assumption is too strong. Lots of demographical, economic and social factors have significant influence on propensity for searching work, labour force participation rate and fertility, so even in the closed economy case statements about constancy of labour force growth rate are impossible to defend, especially when we notice, that there are some problems with labour force measurement. In our opinion it is necessary to consider a bounded periodic function with value in [-1; 1] interval.

Similar problem arises when we are trying to treat labour force growth rate as a function of capital-labour ratio. Again, in calculating the derivative of right-hand-side there appear some ingredients, which may have influence on sign of right-hand-side expression.

Mentioned cases, which in theoretical models connected with Solow's model may be considered, have low or great influence on stability of stationary point. It is important to notice, that knowledge of production function and processes which shape reasonable economy's parameters (like investment rate or coefficient of depreciation of capital assets) give us possibility to find an aswer for a question: will economy, which comes to equlibrium point, stay in it's nearest neighbourhood or maybe that point is unstable and external shocks push it up irretrievably. Solow in his paper gave an example of situation, when unstable solution exists. Existing of solution gives us a possiblity to keep economy near this stable point and knowledge of its properties give important information about character of limit situation for studied economy.

Empirical verification of stability of Solow's model seems arguable. In our opinion the problem is not only with the data, which are still scarce, but also with quality of the available statistical data. Specific features of functioning Polish economy cause that Solow's model may be too general to describe economic variables trajectories. It could be difficult, but necessary to rectify it by adding more specific assumptions and considering more flexible and extended form of growth model.

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