DYNAMIC ECONOMETRIC MODELS
Vol. 6 – Nicolaus Copernicus University – Toruń – 2004

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Wavelet vs. Spectral Analysis of an Economic Process

1. Introduction

Spectral analysis, by which we understand Fourier analysis, makes it possible to decompose a process into constituent sinusoids of different frequencies. Another way to think of Fourier analysis is as a technique for transforming our view of the process from a time-based one to a frequency-based one. However, Fourier analysis has a serious drawback, which consists in loosing time information. When looking at a Fourier transform of a process, it is impossible to tell when a particular event took place. If characteristics of a signal do not change much over time – that is, if it is a stationary signal – this drawback is not an obstacle in a precise analysis of the process. The situation changes dramatically when the signal contains a trend or some transitory characteristics. In an effort to correct deficiencies of Fourier analysis several solutions were proposed. One of them is to adapt the Fourier transform to analyse only a small part of the signal at a time. This adaptation is called Short-Time or Windowed Fourier Transform (STFT or WFT). This transform enables us to view our process in two domains simultaneously, being a sort of compromise between time- and frequency-based analysis. However, this approach has a drawback connected with the same size of a time window for all frequencies, while many economic processes require a more flexible approach – one where we can vary the window size as to analyse long-term movements with larger windows and short-term fluctuations with shorter windows.

Wavelet analysis represents the next logical step in frequency-domain analyses and is a kind of windowing technique with variable-sized regions. A wavelet is – in simplest words – a “small wave”, or – being more precise –

[*] The author acknowledges the support of the Polish Foundation for Science under a scholarship for young scientists in 2004.

a waveform of effectively limited duration and an average value of zero. Comparing sine waves, which are the basis of Fourier analysis, with wavelets one should notice that sinusoids extend from $-\infty$ to $+\infty$ and are regular, while wavelets tend to be asymmetric and irregular (see: Fig. 1).

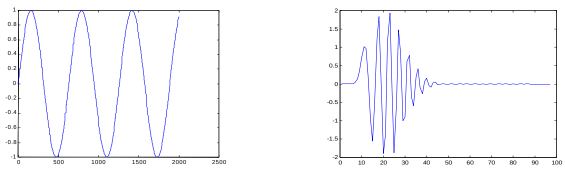


Fig. 1. Sinusoid and db50 (Daubechies 50) wavelet

Wavelet analysis consists in breaking up a signal into shifted and scaled versions of the original (or mother) wavelet. Several filtering methods have been proposed to extract components in a time series, ranging from simple moving averages and least square method to more sophisticated spectral methods, Kalman filters or neural network filters. The main advantage of wavelet analysis is a precise description of the local features of a signal. In contrast to WFT, which is not able to detect events taking place in the range of a time window, wavelet analysis becomes a tool for analysis of non-stationary processes or processes with transient characteristics, which are the results of changing parameters and (or) the non-linearity of underlying mechanisms. This kind of signal processing technique has found use in such differing disciplines as communication, geophysics or medicine, but starts also to find its place in modern finance and economics. This analysis is in some cases complementary to other existing techniques like correlation and spectral analysis, but there are also authors who believe that wavelet filtering provides insight into the dynamics of economic time series beyond that of current methodology and is capable of revealing aspects of data that other time series techniques miss, aspects like breakdown points, discontinuities in higher derivatives, and self-similarity.

Wavelet analysis produces a time-scale view of a signal. The notion of a scale replaces the notion of frequency, so that higher scales correspond to the most "stretched" versions of wavelets. There exists a direct correspondence between a low scale and a high frequency and between a high scale and a low frequency wavelet. Thanks to the ability to adjust the scale wavelets enable us to see both the forest and the trees and make it possible to escape Heisenberg's indeterminacy principle – the law that says that one cannot be simultaneously precise in the time and the frequency domain.

Wavelet analysis is a relatively new signal processing technique, though its mathematical underpinnings date back to the work of Joseph Fourier in the nineteenth century, which is the starting point for all frequency-domain analyses. The first mention of the term "wavelet" was in 1910 in a paper of Harr[1]. The theoretical concept of a wavelet in its present form and the fundamentals of wavelet analysis have arisen in the eighties in France and are connected mainly with two names: J. Morlet and Y. Meyer. Research on wavelets became international in 1988 after finding an algorithm of the fast wavelet transform by S. Mallat. A great interest in applying wavelet analysis in signal processing dates back to that time.

In finance and economics wavelet analysis can be used in:

– analysing properties of economic time series and relationships between them at different time-scales (in long- and short-run);
– investigating local and global features of time series with different resolution (with low or high precision);
– identifying structural breaks, outliers, turning points, discontinuities, and volatility clustering;
– identifying seasonality and seasonal adjusting;
– smoothing and separating trends;
– denoising;
– modelling dynamic of non-linear processes with the help of wavelet networks;
– investigating long-memory processes;
– identifying a fractal nature of economic processes.

In this paper we give a brief overview of basic notions and properties underpinning wavelet analysis with emphasis on discrete wavelet transforms, multiresolution analysis and scalograms. Moreover there are presented examples of economic applications of this technique in identifying periodic components of a time series, smoothing and investigating causal relationships between series according to different scales. So far wavelet methods have not been used extensively in economic time series analysis.

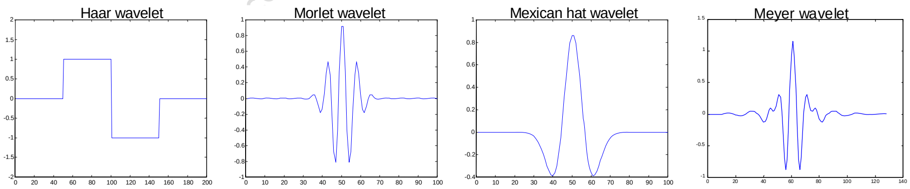


Fig. 2. Examples of wavelets

[1] Haar (1910), Zur Theorie der Orthogonalen Funktionensysteme, *Mathematische Annalen*, 69, 331–371.

2. Basic definitions and properties

A basis of wavelet analysis is a wavelet transform, which – similarly to the Fourier transform – can be continuous or discrete. Let us consider a real-valued function $\psi(\cdot)$ satisfying two basic properties

$$\begin{aligned} &\int_{-\infty}^{\infty} \psi(u)\,du = 0, \\ &\int_{-\infty}^{\infty} \psi^2(u)\,du = 1. \end{aligned} \tag{1}$$

We will refer to this function as a mother wavelet. The continuous wavelet transform (CWT) of function $x(\cdot)$ is

$$W(\lambda,t) = \int_{-\infty}^{\infty} \psi_{\lambda,t}(u)\,x(u)\,du\,, \quad \text{where} \quad \psi_{\lambda,t}(u) = \frac{1}{\sqrt{\lambda}}\psi\left(\frac{u-t}{\lambda}\right), \quad \lambda > 0\,.$$

As a result of applying the continuous wavelet transform we obtain a set of wavelet transform coefficients, which depend on scale λ and time *t*. This set is an equivalent representation of function $x(\cdot)$.

Let $x = (x_0, x_1, \ldots, x_{N-1})'$ be a data vector of length $N = 2^n$. For $j = 0,1,\ldots,n-1$ and $k = 0,1,\ldots,2^j - 1$ we define the discrete wavelet transform (DWT) of vector *x* with respect to $\psi(\cdot)$ as

$$W_{j,k} = \sum_{n=0}^{N-1} x_n \psi_{j,k}\left(\frac{n}{N}\right), \tag{2}$$

where $\psi_{j,k}(\cdot)$ are scaled and shifted versions of the mother wavelet, i.e.

$$\psi_{j,k}(t) = 2^{-j/2}\psi\left(2^{-j}t - k\right). \tag{3}$$

For certain types of the mother wavelet (for example for the Haar wavelet or the wavelet family introduced by I. Daubachies – see Figures above) the set (3) constitutes an orthonormal basis. DWT operates on scales, which are powers of 2 – these are the so-called dyadic scales. Dyadic are also shifts of the form $k \cdot 2^j$. Although DWT can be defined without referring to CWT, we will treat it as a discretisation of the continuous wavelet transform, obtained as a result of critical sampling of CWT. The sampling is critical in the sense, that it gives a minimal number of wavelet transform coefficients, which preserve all the information about the underlying signal.

The discrete wavelet transform is closely related to the so-called multiresolution analysis, which was introduced by Mallat in the end of the eighties. This analysis consists in a multiple-level representation of a signal, where at each level the signal is decomposed into two components: an approximation and a detail. At each consecutive level the approximation from the previous decomposition is represented again as a sum of a subsequent approximation and a detail. The approximations are the high-scale, low-frequency components of the signal, whereas the details are the low-scale, high-frequency components. Proceeding in this way any square-integrable function can be represented with any accuracy as a sequence of details so that when the number of representation levels goes to infinity, the error of this approximation tends to zero.

Technically, a multiresolution analysis projects a function on a set of closed functional subspaces of consecutive approximations:

$$\ldots \subset V_2 \subset V_1 \subset V_0 \subset V_{-1} \ldots . \tag{4}$$

Furthermore, since the subspaces are nested, one can represent V_{j-1} as the direct sum of the coarsely approximated subspace V_j and its orthogonal complement W_j:

$$V_{j-1} = V_j \oplus W_j , \tag{5}$$

W_j represents details of a signal apparent at the level of scale 2^j, which do not appeal for less precise scales. One can think of the subspaces V_j as different levels of magnification, revealing more and more detail. We presume additionally that the subspaces V_j are self-similar, i.e.

$$f(t) \in V_j \Leftrightarrow f(2^j t) \in V_0 , \tag{6}$$

and invariant relative to shifts of the form $k \cdot 2^j$. We assume also that there exists a function $\varphi \in V_0$ such that the set $\{\varphi_{j,n}; n \in \mathbf{Z}\}$ is an orthonormal basis in the subspace V_j, where $\varphi_{j,n} = 2^{-j/2}\varphi(2^{-j}t - n)$. The function φ is the so-called father wavelet (scaling function). Looking for an orthonormal basis in the subspace W_0 one should define

$$\psi(t) = \sqrt{2}\sum_k g_k \varphi(2t - k) . \tag{7}$$

Moreover we have also

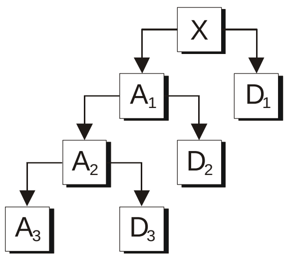
$$\varphi(t) = \sqrt{2}\sum_k h_k \varphi(2t - k) . \tag{8}$$

Equations (7) and (8) are called scaling equations or dilation equations and define sequences $\{g_k\}$ and $\{h_k\}$, which are impulse responses of two filters:

a high-pass filter and a low-pass filter accordingly. The two filters are called quadrature mirror filters. Applying them to an initial signal or its approximation at a given resolution level we get a decomposition of the signal into an approximation (the low-pass filter) and a detail (the high-pass filter). If we denote by G and H the transfer functions of the quadrature mirror filters, a wavelet decomposition of a signal x, consisting of 2^n elements, has the form

$$W = [Gx, GHx, GH^2x, \dots, GH^{n-1}x, GH^n x, H^n x]. \tag{9}$$

Denoting approximations by $A_1 = Hx$, $A_2 = H^2 x$ etc., and details by $D_1 = Gx$, $D_2 = GHx$, $D_3 = GH^2 x$ etc., a wavelet decomposition can be depicted on the following diagram.



$$\begin{aligned} x &= A_1 + D_1 = \\ &= A_2 + D_1 + D_2 = \\ &= A_3 + D_1 + D_2 + D_3 \end{aligned}$$

Fig. 3. Multiresolution analysis

Multiresolution analysis enables showing a signal as a sequence of details, going from fine to coarse. We have to add that the sequence of details (9) given in an opposite order constitutes the discrete wavelet transform (2).

One of the most useful tools in wavelet analysis is the scalogram, which is a counterpart of the periodogram. The energy of the discrete wavelet transform at level j ($j = 0, 1, \dots, n-1$) is defined as

$$E(j) = \sum_{k=0}^{2^{j-1}} W_{j,k}^2 . \tag{10}$$

The scalogram of the data is the vector of energies

$$(A_n^2, E(0), E(1), \dots, E(n-1)). \tag{11}$$

Since for vector $v = (v_1, \dots, v_n)$

$$\|v\|^2 = \sum_{i=1}^{n} v_i^2 ,$$

the scalogram of the discrete wavelet transform can be written in an equivalent form as

$$\left(\left\|H^{n}x\right\|^{2},\left\|GH^{n-1}x\right\|^{2},\ldots,\left\|GH^{2}x\right\|^{2},\left\|GHx\right\|^{2},\left\|Gx\right\|^{2}\right). \tag{12}$$

If an extreme value at a high level j exists in a scalogram, it is an evidence that there are high frequency periodical fluctuations in the data, while if it is at a low level – the periodical oscillations take place for low frequencies.

3. Examples of economic applications

As we mentioned in the Introduction, wavelets are a relatively new way of analysing time series, but in many aspects they are a synthesis of old ideas with a new elegant mathematical approach and an efficient computational algorithm. Wavelets found use in virtually all applications that were previously based on Fourier analysis, but manage also to solve problems for which little progress has been made prior to the introduction of wavelets. They are particularly useful in analysing processes with deterministic or stochastic trends, varying seasonality, structural breaks or outliers. It is known that many economic phenomena – for example economic activity with business cycles – do not follow a strict periodicity. Additionally economic processes have often varying structures and trends. All that causes that wavelet analysis seems to have great potential usefulness in econometrics.

Two properties of wavelets are particularly useful in analysing economic time series: (i) since the base scale includes any non-stationary component, the data need not be detrended or differenced prior to the analysis; (ii) the non-parametric nature of wavelets takes care of a potential non-nonlinear relationship without losing information. Another advantage of wavelets consists in this, that economic actions and decision-making take place at different scales, i.e. they depend on a time horizon. Economists often emphasise the importance of discerning between long-run and short-run behaviour. As Ramsey and Lampart (1998) notice, the classics of economics (J. Hicks for example) saw the necessity to distinguish more time horizons and only pedagogical regards decided that these two periods have been popularised. Wavelets with multiresolution analysis offer the possibility of going beyond this simplifying dichotomy by decomposing time series into several layers of orthogonal scales. The scales can be analysed individually and compered with other series.

In what follows two examples of economic applications of the wavelet decomposition are presented. In the first example the relationship between cash and futures markets in Poland is examined at different levels of resolution. In the study daily quotations of two indices during the period 17.05.1999–20.06.2003 are used: WIG20 from the Polish spot market and synthetic index FW20 from the corresponding futures market (1024 daily observations). A source of current data for the synthetic paper FW20 is a quotation of the futures contract FW20XX with the largest number of transactions. The aim of the analysis was examination of lead-lag relations between quotations in spot and futures markets and answering the question: On which market does the price form? Prices in these two markets are contemporaneously related according to the cost-of-carry model in the theory of finance, in which the "fair value" of a futures contract is equal to the "fair value" of the underlying spot index, plus the cost of carrying the spot index for the duration of the contract. Empirical studies, however, document the existence of some correlation patterns between these two series, which show that futures prices lead spot prices within short time horizons (see for example Bruzda, Wiśniewska, 2002).

In the second example much shorter data series of base inflation indicators and increases of money supply M0 were used (64 monthly observations). The series covered the period 01.1998–04.2003[2]. Here also we try to answer the question: What is the character of the causal relationship decomposed according to time scales?

After a preliminary analysis it turned out that the processes WIG20 and FW20 are integrated of order 1 and that there exists a long-term cointegrating relationship between them[3], whereas the seasonally adjusted inflation indicators and differenced money supply measure M0 are stationary. In what follows non-seasonally adjusted series were used. In Table 1 the estimation output for a vector error correction model for WIG20 and FW20 is included.

Table 1. Estimation output for the VEC model for FW20 and WIG20

Cointegrating equation		
$FW20 - \underset{(\pm 0.02710)}{1.200382} WIG20 + 249.9476 = 0$		
VEC model		
	D(FW20)	D(WIG20)
Error correction	-0.019156 (±0.01959)	0.020774 (±0.01917)
D(FW20(-1))	-0.101135 (±0.07519)	-0.013362 (±0.07357)
D(WIG20(-1))	0.081826 (±0.07729)	-0.003324 (±0.07562)
Const	-0.272333 (±0.88481)	-0.228742 (±0.86572)
R^2	0.0031	0.0016

Source: Author's calculations.

According to Granger's theory the existence of cointegration indicates a causal relationship at least in one direction. However in our case the direction

[2] The data were taken from electronic databases at http://www.nbp.pl and http://bossa.pl/notowania/daneatech/omegasupercharts.

[3] The trace statistic in the Johansen test indicates the existence of a cointegrating vector at the significance level $\alpha = 1\%$.

of the causal relationship between WIG20 and FW20 is hard to determine, because the majority of parameters in the VEC model is insignificant. Granger causality tests provided similar results, indicating the lack of any causal relationship in both directions with the lag length[4] of 2. That is why in further analysis the processes have been decomposed according to different scales in a 6-level wavelet decomposition, where the maximal level of representation was $n = \log_2 1024 = 10$. Results of the decomposition for WIG20 are given in Fig. 4 and results of pairwise Granger causality tests between the approximations (A6) and the details (D1–D6) of the processes WIG20 and FW20 are included in Table 2. The results of Granger tests for decomposed processes are much more informative than initial results. It turns out that for long-term fluctuations (above $2^5 = 32$ observations) the causal relationship runs in two directions, whereas for short-term movements (first and second level of detail) futures prices lead spot prices. The causal relationship and the flow of information in the short-run is from the futures market to the spot market. Additionally charts in Fig. 4 enable observing the quotations of WIG20 with different resolutions (A1–A6), where the highest level of approximation gives the best smoothing of the series, and the plot of continuous wavelet transform can be an indicator of a fractal nature of the process. It is worth noting that the wavelet decomposition can be also useful in forecasting the two processes, because often it is easier to forecast components of a time series than the whole series itself.

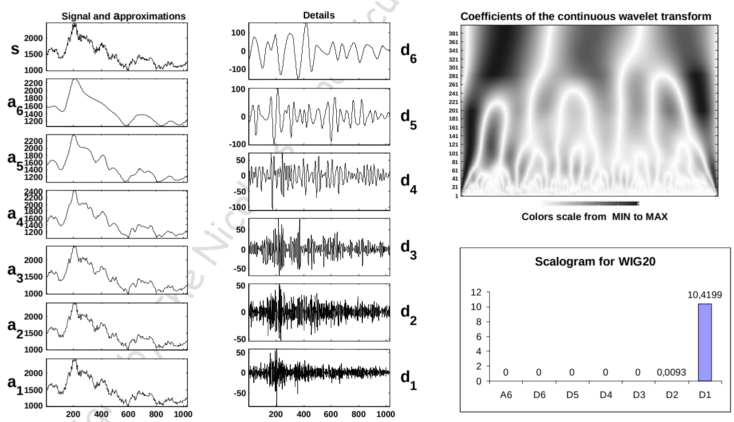


Fig. 4. Multiresolution analysis for WIG20. Scales 1–6 approximately correspond to periods 2–4, 4–8, 8–16, 16–32, 32–64, 64–128

[4] Because of nonstationarity of the variables results of the Granger causality tests should be treated with cautious. These results are available upon request.

Table 2. Results of pairwise causality tests for decomposed processes WIG20 and FW20

Null hypothesis	F statistics	p-value
A6–FW20 does not Granger cause A6–WIG20	77.3345	0.0000
A6–WIG20 does not Granger cause A6–FW20	84.4470	0.0000
D6–FW20 does not Granger cause D6–WIG20	57.4774	0.0000
D6–WIG20 does not Granger cause D6–FW20	12.2121	0.0000
D5–FW20 does not Granger cause D5–WIG20	18.3393	0.0000
D5–WIG20 does not Granger cause D5–FW20	7.8326	0.0004
D4–FW20 does not Granger cause D4–WIG20	0.4063	0.6662
D4–WIG20 does not Granger cause D4–FW20	1.1093	0.3302
D3–FW20 does not Granger cause D3–WIG20	2.2758	0.1032
D3–WIG20 does not Granger cause D3–FW20	6.6236	0.0014
D2–FW20 does not Granger cause D2–WIG20	3.5203	0.0300
D2–WIG20 does not Granger cause D2–FW20	0.7229	0.4910
D1–FW20 does not Granger cause D1–WIG20	2.9525	0.0527
D1–WIG20 does not Granger cause D1–FW20	2.2721	0.1036

Lags length in testing equations was fixed to 2.

Source: Author's calculations.

In the second example a 4-level multiresolution analysis was performed, where the maximal resolution level was $n = \log_2 64 = 6$. Results of the decomposition are given in Fig. 5. One can observe that for both series periodic fluctuations have been mainly taken up by details at the third level (D3), which corresponds to movements of periods 8–16 and includes also seasonal fluctuations. It can be evidence of the seasonal character of inflation and money supply. For the decomposed series causality analysis has been performed in the same way as in the previous example. Pairwise Granger causality tests (see: Table 3) indicate that for medium-term fluctuations increases of money supply cause inflation, while in the long-run causation runs in both directions.

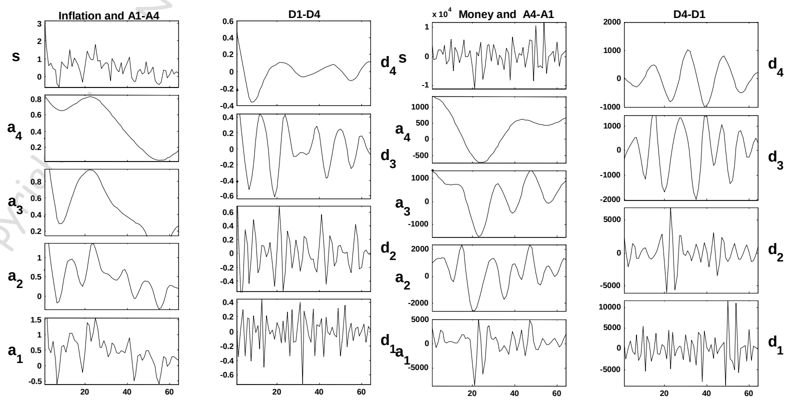


Fig. 5. Multiresolution analysis for inflation index and money supply increments. Scales 1–4 approximately correspond to periods 2–4, 4–8, 8–16, 16–32

Table 3. Results of pairwise causality tests for decomposed processes INF and ΔM0

Null hypothesis	F statistics	p-value
A4–INF does not Granger cause A4–ΔM0	24.0976	0.0000
A4–ΔM0 does not Granger cause A4–INF	88.7890	0.0000
D4–INF does not Granger cause D4–ΔM0	8.3308	0.0007
D4–ΔM0 does not Granger cause D4–INF	1.8993	0.1590
D3–INF does not Granger cause D3–ΔM0	1.7102	0.1900
D3–ΔM0 does not Granger cause D3–INF	3.0417	0.0556
D2–INF does not Granger cause D2–ΔM0	2.0721	0.1353
D2–ΔM0 does not Granger cause D2–INF	0.1740	0.8408
D1–INF does not Granger cause D1–ΔM0	1.1370	0.3280
D1–ΔM0 does not Granger cause D1–INF	1.1824	0.3140

Lags length in testing equations was fixed to 2.
Source: Author's calculations.

4. Conclusions

Wavelet analysis can be treated as frequency-domain analysis for nonstationary and non-linear processes providing insight into the dynamics of economic time series beyond that of current methodology. This kind of time series techniques is capable of revealing such aspects of data like breakdown points, discontinuities or self-similarity and becomes a tool for the analysis of processes with transient characteristics, which are results of changing parameters or non-linearity of underlying mechanisms. The wavelet decomposition is a kind of filtration, which decomposes a time series according to different scales and makes it possible to analyse the series individually and compare with other series. Decomposing a time series into different scales may reveal details that can be interpreted on theoretical grounds as well as be used to improve forecasting accuracy. The wavelet technique, however, is capable of handling stationary as well as nonstationary processes. Thanks to this the wavelet decomposition can be a method of investigating long-run economic relationships and an alternative for cointegration analysis or the concept of co-trending of economic processes. Additionally, if we relax the assumption of linearity of a long-term relationship, the wavelet decomposition combined with non-linear causality tests (like Brooks and Hinich or Hiemstra and Jones tests) constitutes also an alternative for the concepts of non-linear cointegration (see: Granger and Hallman, 1991) and non-linear co-trending (see: Bierens, 2000).

References

Ariño, M. A., Morettin, P., Vidakovic, B. (1995), Wavelet Scalograms and Their Application in Economic Time Series, Institute of Statistics and Decision Sciences, Duke University, *Discussion Paper*, 95–21.

Białasiewicz, J. T. (2000), *Falki i aproksymacje* (*Wavelets and Approximations*), Wydawnictwa Naukowo-Techniczne, Warszawa.

Bierens, H. J. (2000), Nonparametic Nonlinear Co-Trending Analysis, With an Application to Interest and Inflation in the U.S., *Journal of Business and Economic Statistics*, 18, 323–337.

Bruzda, J., Wiśniewska, E. (2002), Badanie zależności pomiędzy cenami terminowymi i cenami spot na przykładzie kontraktów futures na WIG20 (Investigating Dependences Between Spot and Futures Prices on an Example of the Contract FW20), in: ed. W. Tarczyński *Rynek kapitałowy. Skuteczne inwestowanie* (*Capital market. Effective investing*).

Daubechies, I. (1992), *Ten Lectures on Wavelets*, Capital City Press, Montpelier,

Gençay, R. F., Selçuk, F., Whitcher, B. (2002), *An Introduction to Wavelets and Other Filtering Methods in Finance and Economics*, Academic Press, San Diego.

Granger, C. W. J., Hallman, J. J. (1991), Long-memory Processes With Attractors. *Oxford Bulletin of Economics and Statistics*, 53, 11–26.

Lin, S.–J., Stevenson, M. (2001), Wavelet Analysis of the Cost-of-Carry Model, *Studies in Nonlinear Dynamics and Econometrics*, 5(1), 87–102.

Misiti, M., Misiti, Y., Oppenheim, G., Poggi, J.–M. (1996), *Wavelet Toolbox For Use with MATLAB®*, The MathWorks.

Percival, D. B., Walden, A. T. (2000*), Wavelet Methods for Time Series Analysis*, Cambridge University Press, Cambridge.

Ramsey, J. B. (1999), The Contribution of Wavelets to the Analysis of Economic and Financial Data, *Philosophical Transactions of the Royal Society of London*, Series A, 357, 2593–2606.

Ramsey, J. B., Lampart, C. (1998), The Decomposition of Economic Relationships by Time Scale Using Wavelets: Expenditure and Income, *Studies in Nonlinear Dynamics and Econometrics*, 3(1), 23–42.

Schleicher, Ch. (2002), An Introduction to Wavelets for Economists, *Working Paper*, 2002–3, Bank of Canada.

Talaga, L., Zieliński, Z. (1986), *Analiza spektralna w modelowaniu ekonometrycznym* (*Spectral Analysis in Econometric Modelling*), PWN, Warszawa.