DYNAMIC ECONOMETRIC MODELS
Vol. 6 – Nicolaus Copernicus University – Toruń – 2004

Czesław Domański
University of Lodz, Institute of Econometrics and Statistics

Application of Runs of Signs Tests in the Statistical Process Control

1. Introduction

The basic task of the statistical process control is to isolate the results of measurements which indicate the existence of interference in a stable run.

Let us assume that random variables disturbing the stable run of the process have an additive character, that is they are added to a variable corresponding with the stable (regulated) process.

The sample sampled at moment t corresponds with observations of a certain random variable $Y(t)$

$$Y_t = X_0 + \sum_{i=1}^{k} I_i(t) X_i, \tag{1}$$

where

$x_i \approx F_i(\mu_i, \sigma_i^2)$

and the index function $I_i(t)$ has the following form

$$I_i(t) = \begin{cases} 1 & z\ p-wem\ p_i \\ 0 & z\ p-wem\ (1-p_i) \end{cases}. \tag{2}$$

If the considered model (1) is correct and we have k possible determined reasons of maladjustment then, in the given sample one of 2^k possible combinations of random variables corresponding with these reasons occurs. Let us notice that extreme possibilities from among all 2^k combinations are as follows:

– not occurring any of maladjustment reasons $(I_i(t) = 0 \; dla \; i = 1,...,k)$ for all $i = 1,...,k$,
– occurring all of maladjustment reasons $(I_i(t) = 1$ for all $i = 1,...,k$).

We assume that samples are sampled seldom enough so that the form of the variable Y for one sample is independent of its form for another sample. What is more, we assume that every index function I_i preserves the constant value (0 or 1) during one sample sampling which means that the time of the sample sampling is short enough, so that the sample consists of observations of the only one random variable Y (see Thompson, Koronacki (1994)).

2. Distributions of the number and the length of monotonic runs

Let $y_1, y_2, ..., y_n$ denote the sequence of time ordered observations made on the variable Y_t

Considering the sequence of signs

$$\operatorname{sgn}(y_2 - y_1), \operatorname{sgn}(y_3 - y_2), ..., \operatorname{sgn}(y_n - y_{n-1}), \tag{3}$$

where:

$$\operatorname{sgn}\; y = \begin{cases} +, & when \quad y > 0, \\ -, & when \quad y < 0, \end{cases} \tag{4}$$

we can get the so called monotonic runs[1].

Asymptotic distributions of monotonic runs have been found by H. Levene (1953) and J. Wolfowitz (1944). J Wolfowitz showed that the limiting distribution of the run of signs length, assuming that the trend does not exist, is the Poisson exponential distribution. The proof of his theorem is used for the distribution of runs with the length exactly amounting to *r*. However, assumptions made while its derivation can be used for the run distribution with the length *r* or bigger. P. S. Olmsted (1958) indicated that the compatibility of the distribution of the runs of signs length with the Poisson distribution is of the order 0.0001 for $r \geq 6$, 0.001 for $r \geq 5$, 0.01 for $r \geq 4$, and 0.1 for $r \geq 3$, if $n \geq 14$. Hence, for $r \geq 5$ the probability[2] of getting, with *n* observations, at least one run of the length of at least *r* elements is equal to:

$$P\{L_r^{'} \geq 1\} = 1 - e^{-E(L_r^{'})}, \tag{5}$$

[1] Function sgn *y* is not defined for *y=0,* yet the probability of the occurrence of *y* =0 event under the assumption that the random variable is continuous is equal to zero.

[2] Critical values for the runs test based on the length of runs can be determined on the basis of the Poisson exponential distribution. For $r \geq 5$ the relation $D^2(L_r^{'}) = E(L_r^{'})$ occurs.

where:

$$E(L_r^{'}) = \frac{2[(n-1)(r+1)+1]}{(r+2)!} \quad \left(for \quad \frac{n}{2} \le r \le n-1 \right). \tag{6}$$

E. S. Edigington (1961) gave the recurrent formula which enabled him to prepare tables of the distribution of runs of signs "+" or "-". Based on this table, the critical values for the test based on the number of monotonic runs were indicated. They are included in Table 1. This table gives such greatest total numbers l_α for which $P\{L \le l_\alpha\} \le \alpha$, if $\alpha < 0{,}5$ and such smallest total numbers l_α, for which $P\{L \le l_\alpha\}$, if $\alpha > 0{,}5$ for ten probabilities α (0.005, 0.01, 0.025, 0.05, 0.10, 0.90, 0.95, 0.975, 0.99, 0.995). Placed quantiles of the variable L significantly simplify the usage of tests of runs based on the number of monotonic runs.

E. S. Edigington shows that the random variable L for $n > 25$ has the asymptotic normal distribution of parameters given by the following formulas:

$$E(L) = \frac{2n-1}{3}, \tag{7}$$

$$D^2(L) = \frac{16n-29}{90}. \tag{8}$$

The expected value of the variable L_r, indicating the number of runs of the length r, takes the form of:

$$E(L_r) = \frac{2}{(r+3)!}\left[n(r^2+3r+1)-(r^3+3r^2-r-4)\right], \tag{9}$$

for $r \le n-2$,

and the variance of the variable $L_r^{'}$ is expressed by the following formula:

$$D^2(L_r^{'}) = \left[E(L_r^{'})\right]\left[1-\frac{3}{r!}-\frac{r!}{(2r)!}+\frac{1}{2}\left(\frac{1}{(r!)^2}+\frac{1}{(2r)!}\right)\right]. \tag{10}$$

For practical reasons, formulas for variances of variables L_r and $L_r^{'}$ for $r \le 3$ can be applied:

$$D^2(L_1) = \frac{305n-347}{720}, \tag{11}$$

$$D^2(L_2) = \frac{51106n-73\,859}{453\,600}, \tag{12}$$

$$D^2(L_1^{'}) = \frac{16n-29}{90}, \tag{13}$$

$$D^2(L_2^{'}) = \frac{57n-43}{720}, \tag{14}$$

$$D^2(L_3^{'}) = \frac{21\,496n - 51\,269}{453\,600}. \tag{15}$$

Runs of signs of upwards and downwards deviations, that is so called monotonic runs are of great importance in econometric research as well as in the statistical quality inspection. Their greatest advantage is that it becomes obvious at once that the sum of the length of all runs is *(n*–1*)* where *n* is the number of observations.

3. Tests based on the general number of the runs of signs

Let $y_1, y_2, ..., y_n$ denote time ordered realizations of random variables Y_t. On the ground of this sequence it is necessary to verify the hypothesis H_0 according to which expected values of these variables are identical, that is the investigated sequence does not contain a trend. Based on the sample results we create a sequence of signs $\text{sgn}(y_{i+1} - y_i)$. Next, we determine the gauge value of this test – the statistic *l* which is the general number of runs of signs "+" and "–". If the sequence $\{x_i\}$ is random then a relatively large number of short runs of signs "+" and " –"should be expected. However, if there is a trend in the sequence then we can expect many long runs while these runs will generally consist of signs "+" if the trend is increasing and signs "–"if the trend is decreasing. Therefore, the critical region for this test is built on the left side, that is if a relation $l \le l_\alpha$, proceeds, the null hypothesis should be rejected. Critical values can be read from Table 1. The variable *L* has the asymptotically normal distribution *for n*>25 with parameters given by formulas (7) and (8).

4. Moor–Wallis test

The Moor–Wallis test is based on the number of signs "+"which is defined by the symbol z^+. The hypothesis H_0 is rejected for the alternative one that a decreasing trend occurs if $z^+ < z_\alpha^{+'}$ where:

$$z_\alpha^{+'} = \frac{n}{2} - \left| Z_\alpha^+ - \frac{n-1}{2} \right|, \tag{16}$$

$Z_\alpha^{+'}$ can be read from Table 2 for adequate *n* and α.

Analogously, the hypothesis H_0 is rejected for the alternative one that an increasing trend occurs if $z^+ > z_\alpha^+$, where:

$$z_\alpha^+ = \frac{n-2}{2} + \left| Z_\alpha^+ - \frac{n-1}{2} \right| . \tag{17}$$

We assume a hypothesis that an increasing or decreasing trend occurs if the following inequalities proceed:

$$z^+ > z_{\alpha/2}^{+'} . \tag{18}$$

If $n \geq 13$ and

$$\frac{1}{3n} \leq \alpha \leq 1 - \frac{1}{3n} \tag{19}$$

then z_α^+ and $z_\alpha^{+'}$ are denoted based on the following formulas:

$$z_\alpha^{+'} = \frac{1}{2} + \frac{n-1}{2} - u_\alpha \sqrt{\frac{n+1}{12}} \tag{20}$$

and

$$z_\alpha^+ = -\frac{1}{2} + \frac{n-1}{2} + u_\alpha \sqrt{\frac{n+1}{12}}, \tag{21}$$

where u_α is a quantile of α type of the normal distribution $N(0,1)$, while we must assume such α so that these values are integer.

5. χ^2 test based on expected and observed monotonic runs of various length

On the ground of formulas (6) and (9) we build a theoretical distribution of the number of runs of various lengths and the compatibility of the observed distribution with the theoretical one is verified by means of χ^2 test.

F. Moore and W. Wallis (1943) proposed the following gauge for this test:

$$\chi^2 = \frac{(l_1 - e_1)^2}{e_1} + \frac{(l_2 - e_2)^2}{e_2} + \frac{(l_3^{'} - e_3^{'})}{e_3} , \tag{22}$$

where:

l_1 – the number of runs of the length 1,

l_2 – the number of runs of the length 2,

$l_3^{'}$ – the number of runs of the length bigger than 2,

and $e_1, e_2, e_3^{'}$ are defined by formulas:

$$e_1 = E(L_1) = \frac{5n+1}{12}, \tag{23}$$

$$e_2 = E(L_2) = \frac{11n-14}{60}, \quad (24)$$

$$e_3^{'} = E(L_3^{'}) = \frac{4n-11}{60}. \quad (25)$$

The hypothesis H_0 is rejected for the alternative one, that a trend or cycles occur if $\chi^2 > \chi_\alpha^{'2}$.

$$\chi^2 = \begin{cases} \frac{7}{6}\chi_\alpha^2 \text{ when } \chi^2 \le 6{,}3 \\ \chi_\alpha^2 \text{ when } \chi^2 > 6{,}3 \end{cases}$$

while in the first case χ_α^2 is read for 2 degrees of freedom and in the second one for 2.5 degrees of freedom (see Domański, Pruska (2000)).

6. χ^2 test based on the number of runs of signs

Another χ^2 test is based on expected and observed runs of various lengths. The gauge of this test is the statistic:

$$\chi^2 = \frac{\left[z^+ - E(Z^+)\right]^2}{D^2(Z^+)} + \frac{\left[l - E(L)\right]^2}{D^2(L)}, \quad (23)$$

where:

z^+ – the number of signs ”+”,

l – the general number of runs of signs,

and expected values and variances are defined by formulas:

$$E(Z^+) = \frac{n-1}{2}, \quad (24)$$

$$E(L) = \frac{2n-1}{3}, \quad (25)$$

$$D^2(Z^+) = \frac{n+1}{2}, \quad (26)$$

$$D^2(L) = 0{,}178n - 0{,}323. \quad (27)$$

The hypothesis H_0 is rejected for the alternative one that a trend or cycles occur if $\chi^2 > \chi_\alpha^2$ for 2 degrees of freedom.

7. Final remarks

The problem of the randomness occurs in the quality inspection of produced goods very often. Presented randomness tests fit especially the quality inspection. However, they seem to have the following shared features:

1. They are based on runs belonging to the sequence: $y_1, y_2, ..., y_n$,
2. Testing procedure is invariant with respect to typological transformations of the abscise axis, that is the test gives good results if variables $y_1, y_2, ..., y_n$, are replaced with $y_1^{'}, y_2^{'}, ..., y_n^{'}$ where $y^{'} = f(y)$ and $f(t)$ is optional continuous and strictly monotonic function t.
3. The size of the critical region, that is probabilities of the rejection of the randomness hypothesis if it is true does not depend on the shared cumulative distribution function $F(y)$ of variables $y_1, y_2, ..., y_n$.

   Condition 3 is fulfilled if condition 2 proceeds and if $F(y)$ is continuous.

The runs theory as the supportive tool of the quality inspection has been developing in the following direction:

a) While making decisions concerning the manufacturing process it is necessary to choose such types of runs and statistics which are based on appropriate typical deviations from states related to the statistical control, which engineers consider to be the most realizable. Various manufacturing processes demand various modifications of statistical methods.
b) Theories and procedures based on the Monte Carlo methods which enable us to create a precise definition of the power test functions as well as connections between various testing statistics should be developed.
c) Using statistical methods to estimate the quality of the manufacturing process it is necessary to take into consideration the minimalization of financial losses.

Table 1. Critical values for tests of runs of signs “+” and “–”*

n \ l_α	$l_{0.005}$	$l_{0.01}$	$l_{0.25}$	$l_{0.05}$	$l_{0.10}$	$l_{0.90}$	$l_{0.95}$	$l_{0.75}$	$l_{0.99}$	$l_{0.999}$
2						1	1	1	1	1
3						2	2	2	2	2
4					1	3	3	3	3	3
5			1	1	1	4	4	4	4	4
6	1	1	1	1	2	5	5	5	5	5
7	1	1	2	2	2	6	6	6	6	6
8	1	2	2	2	3	6	7	7	7	7
9	2	2	2	3	3	7	7	8	8	8
10	2	3	3	3	4	8	8	9	9	9
11	3	3	4	4	4	9	9	9	10	10
12	3	4	4	4	5	9	10	10	11	11
13	4	4	5	5	6	10	11	11	11	12
14	4	5	6	6	6	11	11	12	12	12
15	5	5	6	6	7	12	12	13	13	13
16	5	6	6	7	7	12	13	13	14	14
17	6	6	7	7	8	13	14	14	15	15
18	6	7	7	8	8	14	14	15	15	15
19	7	7	8	8	9	15	15	16	16	17
20	7	8	8	9	10	15	16	16	17	17
21	8	8	9	10	10	16	17	17	18	18
22	9	9	10	10	11	17	17	18	19	19
23	9	10	10	11	12	17	18	19	19	19
24	10	10	11	11	12	18	19	19	20	21
25	10	11	11	12	13	19	20	20	21	21

* The table contains, for a few probabilities α, such biggest integer numbers l_α for which $P(L \le l_\alpha) \le \alpha$, if $\alpha < 0.5$ and such smallest ones l_α, for which $P(L \le l_\alpha) \ge \alpha$, if $\alpha > 0.5$. *Source: Author's calculations on the basis of the paper of Edigington (1961).*

Table 2. Values α for the test of the run based on the number of signs “+”

Z_α \ n	3	4	5	6	7	8	9	10	11	12
0	0.167	0.042	0.008							
1			0.225	0.081	0.024	0.006				
2					0.260	0.113	0.042	0.013	0.004	
3							0.285	0.139	0.049	0.022
4									0.303	0.161

Source: Author's calculations on the basis of Walsh (1962).

References

Domański, Cz., Pruska, K. (2000), *Nieklasyczne metody wnioskowania statystycznego* (*Non–classical methods of statistical inference*), PWE, Warszawa.

Edigington, E. S. (1961), Probability table for number of runs of signs of first differences in ordered series, *Journal of the American Statistical Association*, vol. 56, 156–159.

Levene, H. (1953), On the power function of tests of randomness based on runs up and down, *Annals of Mathematical Statistics*, vol. 23, 34–56.

Moore, G. H., Wallis, W. A. (1943), Times series significance tests based on signs of differences, *Journal of the American Statistical Association*, vol. 38, 153–164

Olmsted, P. S. (1958), Runs determined in a sample by an arbitrary cut, *Bell System Technical Journal*, t. 37, 55–82.

Thompson, J. R., Koronacki, J. (1994), *Statystyczne sterowanie procesem. Metoda Deminga etapowej optymalizacji jakości* (*Statistical Process Control. Deming Method of Quality Optimalisation*), Akademicka Oficyna Wyd. PL, Warszawa.

Walsh, J. E. (1962), *Handbook of Nonparametric Statistics,* D. Can Nostrand Co. Inc., Princeton.

Wolfowitz, J. (1944), Asymptotic distribution of runs up and down, *Annals of Mathematical Statistics*, vol. 15, 163–172.