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Jumps Activity and Singularity Spectra for Instruments in the Polish Financial Market[†]

A b s t r a c t. In the paper we try to measure the activity of jumps in returns of some instruments from the Polish financial market. We use Blumenthal-Getoor index β for Lévy processes as a measure of jumps' activity. This allows us to distinguish between processes with rare and sharp jumps and the processes with infinitely-active jump component. We use three different methods. First we use activity signature plots to estimate the activity patterns of jumps. Then we estimate the Blumenthal-Getoor index with Aït-Sahalia and Jacod threshold estimator.Then we use methods based on singularity spectra of Lévy processes. Finally, we compare the results.

K e y w o r d s: Blumenthal-Getoor index, singularity spectrum, Lévy exponential models.

Introduction

The classical models of assets' returns are based on the assumption of the normality of returns. This is for example the case of classical portfolio theory, developed by Markowitz (1952) and Sharpe (1963), and the option pricing formula, derived by Black and Scholes (1973) and by Merton (1973). However it is well-known that the normality assumption does not hold. Many well-established stylized facts about assets' returns contradict this. The observed returns reveal characteristics such as heavy tails, high kurtosis or volatility clustering, which is not consistent with Gaussian distribution[1].

There are several methods to deal with non-normality of returns and to make models better fitted to observed data. The most popular approach is the modeling the conditional volatility with some kind of ARCH/GARCH model.

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[1] The survey of stylized facts concerning assets' returns can be found in (Cont, 2001).

When using non-Gaussian distributions of error terms, such models seem to be well-fitted to the data.

In this work we take another approach, which lately becomes more and more popular, namely we use Lévy processes in the modeling. Although this line of modeling is as old as the seminal paper of Maldenbrot (1963), in which models with stable distributions were proposed, it became more popular at the beginning of XXI century.

Lévy processes can be represented as a sum of continuous diffusion (Wiener process) and discontinuous jumps. Thus, one of the main questions is if there are jumps in the returns of financial instruments and if there are, how much intensive are the jumps. Formally the intensity of jumps of Lévy process is described by Blumenthal-Getoor index. In the paper we try to estimate the value of this index for the suitably-chosen sample of instrument from Polish financial market. We use three different methods of estimation and then we compare the results.

The paper is organized as follows. In the section 1 we present basic facts about Lévy processes and Lévy exponential models of prices. In this section we also introduce the concept of Blumenthal-Getoor index of jumps. In section 2 we try to analyze the type of processes using activity signature plots. In section 3 we estimate Blumenthal-Getoor indexes using threshold estimator. In section 4 we analyze the type of processes using its singularity spectrum. Section 5 concludes.

1. Lévy Processes and Lévy Exponential Models

The Lévy process L is the stochastic process with continuous time that starts at zero (i.e. $L_0 = 0$) and fulfils the following conditions:

1. for any $0 \le t_1 < t_2 \le t_3 < t_4$ the random variables $L_{t_2} - L_{t_1}$ and $L_{t_4} - L_{t_3}$ are independent and the distribution of $L_{t+h} - L_t$ depends only on h and not on t ,
2. the process is stochastically continuous, i.e. for all $t \ge 0$ and all $\varepsilon > 0$

   $$\lim_{h \to 0} P\left(\left|L_{t+h} - L_t\right| > \varepsilon\right) = 0$$

3. the trajectories of the process are *cadlag* (i.e. they are right-continuous with left limits).

We should stress that the second condition does not imply that the process is continuous. In fact the Lévy processes typically have discontinuities (of “jumps”) and some of them are discontinuous at each point t . The condition 2 means only that the process L does not have the jump of arbitrary size ε at any pre-specified moment t .

The Lévy processes can be seen as the extension of Wiener process. In fact if we change the condition 2 and require the process L to be continuous, then the only processes that fulfill the definition are Wiener processes with drift. There is also another connection between Wiener process and Lévy processes. The increments of a Wiener process are normally-distributed, while the increments of Lévy process belong to the family of infinitely-divisible distributions. It is the biggest family of distributions that can serve as limits for the sums of independent variables (see Feller, 1971). Thus, if we believe that the returns of the assets in financial markets result from many independent shock, then the Lévy processes are natural choice as an engine in the model.

There are two fundamental theorems that reveal the structure of the Lévy processes. The first one is the Lévy-Itô decomposition, which states that any Lévy process L can be uniquely represented as the sum of Wiener process with drift, Poisson process and purely discontinuous martingale:

$$L_t = \mu t + \sigma W_t + X_t^l + X_t^s , \tag{1}$$

where μ and $\sigma \geq 0$ are constants, W is a standardized Wiener process, X^l is a compound Poisson process of large jumps and X^s is purely discontinuous martingale (i.e. it is discontinuous at each point t). The process X^s represents small jumps – in every finite interval there are infinitely many of them, but they are very small, so that the process X^s does not explode.

The second theorem characterizes the characteristic function of the Lévy processes. According to Lévy-Khintchine representation the characteristic function of the Lévy process is:

$$E[e^{iuL_t}] = \exp[t\psi(u)] , \tag{2}$$

where the characteristic exponent ψ equals:

$$\psi(u) = i\mu u - \frac{\sigma^2}{2}u^2 + \int_R \left(e^{iux} - 1 - iux 1_{|x|<1} \right) d\nu(x) . \tag{3}$$

The first two terms in the sum (3) are the same as in the characteristic function for the Gaussian distribution. These terms describes continuous part of the process (diffusion). The measure ν (called Lévy measure) describes the jumps of the process L. The value of $\nu(R)$ is the intensity of jumps. If finite, it is the average number of jumps in the unit of time – the process is then said to be “finitely active”. If $v(R) = \infty$, then the process is infinitely active – in any interval the number of jumps is infinite. The values $v((a,b))$ give the relative intensity of jumps with sizes between a and b (i.e. jumps such that $\Delta L_t \equiv L_t - L_{t-} \in (a,b)$). Thus the Lévy measure contains information of both the intensity of jumps and distribution of jumps' sizes. The triple (μ, σ, ν), called "characteristic triple", gives unique characterization of the process.

The measure ν, which contains all information of jumps and its structure, can vary, depending on the type of the process or on the probability distributions of increments of the process. There exists however one synthetic index that divides measures ν into certain categories and gives characteristics of jump behavior. Blumenthal-Getoor index is defined as

$$\beta = \inf \left\{ b > 0 : \int_{-1}^{1} |x|^b \, d\nu(x) < \infty \right\}. \tag{4}$$

The index β takes values in the interval $[0, 2)$. Higher values of β mean that jumps are more intensive and smaller and the process L resemble continuous process. If $\beta = 0$ the process is finitely active. All other values mean that the activity of the process is infinite. In the case $\beta = 0$ the discontinuous part of the process L is the compound Poisson process, i.e. $X^s \equiv 0$ in the decomposition (1).

The models of assets prices driven by Lévy process usually take the form of exponential Lévy models:

$$S_t = S_0 e^{L_t}, \tag{5}$$

where S_t denotes the asset price at time t. The models can be also formulated as stochastic differential equation (as in classis Black-Scholes model):

$$dS_t = S_{t-} dV_t, \tag{6}$$

where V is a Lévy process, whose characteristics can be derived from L[2]. The logarithms of the prices are described by Lévy process L and thus the logarithmic returns are increments of Lévy process. According to (1) the logarithmic price is given by:

$$s_t \equiv \ln S_t = \mu t + \sigma W_t + X_t^l + X_t^s. \tag{7}$$

In the literature one considers also the generalization of (7), assuming that the volatility of continuous part is not constant. Such a model can be specified as follows:

$$s_t = \mu t + \sigma_t W_t + L_t^d, \tag{8}$$

where σ_t is a process or a function representing volatility and L^d is discontinuous part of the process L and represents jumps.

[2] Kallsen (2000) has shown that specifications (5) and (6) are equivalent and gave the formulae how to express V in terms of L and vice versa.

Let us select some time-scale, i.e. choose some frequency in which we sample the process. Suppose that we sample every h units of time (in practice it can be a period from several seconds to one day). The increments of the process at the specified frequency are logarithmic returns of the assets. We denote them by $x_i(h)$:

$$x_i(h) = s_{(i+1)h} - s_{ih} = L_{(i+1)h} - L_{ih} . \tag{9}$$

If there is no ambiguity about frequency we omit brackets and denote returns simply by x_i. In the next three sections we will use data about returns to calculate index β, using three different methods.

2. Estimating Jumps Activity with Activity Signatures

The first method is based on power variation of stochastic process, defined as:

$$V(p,h) = \sum_{i=1}^{N} |x_i(h)|^p . \tag{10}$$

In the case $p = 2$, the $V(2,h)$ is well-known realized volatility, which tends to the quadratic variation of the process, as the sampling frequency tends to infinity. The behavior of $V(p,h)$ in other cases depends on the type of the process. As Barndorff-Nielsen and Shephard (2002) have shown, if the process L contains no jumps (i.e. $L^d \equiv 0$ in (8)), then:

$$\operatorname*{plim}_{h \to 0} h^{1-p/2} V(p,h) = \mu_p \int_0^T |\sigma_s|^p \, ds , \tag{11}$$

where μ_p is appropriate constant. If the process L contains no diffusion part ($\sigma_t \equiv 0$), then as $h \to 0$ the sum (10) diverge for $p < \beta$. On the other hand for $p > \beta$ the sum is convergent:

$$\operatorname*{plim}_{h \to 0} V(p,h) = \sum_{s=0}^{T} |\Delta L_s|^p . \tag{12}$$

If the process contains both diffusion and jump parts, then the following equations holds:

$$\begin{cases} \operatorname*{plim}_{h\to 0} h^{1-p/2} V(p,h) = \mu_p \int_0^T |\sigma_s|^p \, ds, & \text{for } p \in (0,2), \\ \operatorname*{plim}_{h\to 0} V(2,h) = \int_0^T |\sigma_s|^2 \, ds + \sum_{s=0}^{T} |\Delta L_s|^2, & \text{for } p = 2, \\ \operatorname*{plim}_{h\to 0} V(p,h) = \sum_{s=0}^{T} |\Delta L_s|^p, & \text{for } p > 2. \end{cases} \tag{13}$$

Based on these limit behavior of the power variation, Todorov and Tauchen (2009) proposed a qualitative test of the type of the process. They have defined activity signature function $\hat{\beta}(p;k,h)$ as:

$$\hat{\beta}(p;k,h) = \frac{p \ln k}{\ln k + \ln V(p,kh) - \ln V(p,h)} \,. \tag{14}$$

The graph of $\hat{\beta}(p;k,h)$ with respect to p are called “activity signature plot”. If the process is continuous, then for all $p > 0$:

$$\operatorname*{plim}_{h\to 0} \hat{\beta}(p;k,h) = 2 \,. \tag{15}$$

For processes of pure jumps we have:

$$\operatorname*{plim}_{h\to 0} \hat{\beta}(p;k,h) = \begin{cases} \beta, & \text{for } p \in (0,\beta), \\ p, & \text{for } p > \beta. \end{cases} \tag{16}$$

In the case, when the process contains both jumps and diffusion:

$$\operatorname*{plim}_{h\to 0} \hat{\beta}(p;k,h) = \begin{cases} 2, & \text{for } p \in (0,2), \\ p, & \text{for } p \geq \beta. \end{cases} \tag{17}$$

We apply method of activity signatures to several instruments from Polish financial markets. The sample contains of four stocks: two liquid ones (PKN Orlen and KGHM) and two less liquid (Agora and BRE Bank), three stock market indexes (WIG, WIG20 and MWIG40), one future contract (FWIG) and two currencies (euro and US dollar). The sample was chosen as to contain possibly wide range of different instruments. We have used intraday data for the period from the beginning of 2009 to the March of 2011. In the computation we used 10-minutes returns. We have tried some other frequencies and decided that this frequency is high enough to justify the usage of the limits (15)-(17). On the other hand it is not so high, that the market microstructure noise affects the results[3]. In the computations of $\hat{\beta}(p;k,h)$ we took $k = 2$, as in the original work

[3] We have used methods of Zhang, Mykland, Aït-Sahalia (2005) to control the microstructure noise. For sampling period of 10 minutes the difference between “two time scales” estimator of

of Todorov and Tauchen (2009). Thus we have worked with two time scales: 10 minutes and 20 minutes returns.

We do not present results for each instrument, but only show three typical cases. Figure 1 shows activity signature plot for Agora. Such a graph is typical for less-liquid stocks. It represents the case (16) of pure jump process with $\beta = 0$, which means that the prices are driven by compound Poisson processes.

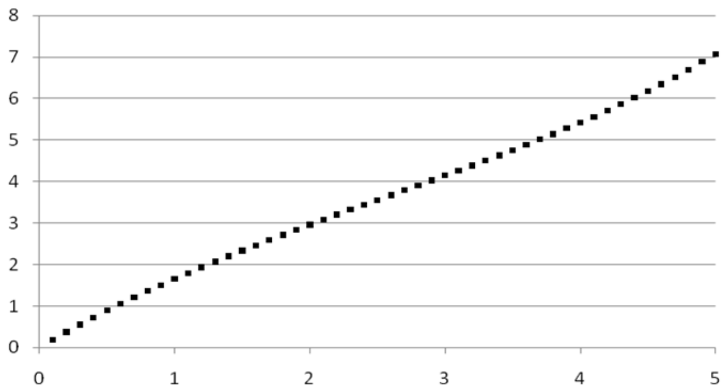

Figure 1. The activity signature plot for Agora (vertical axis – exponents p, horizontal axis – the values of $\hat{\beta}(p;2,h)$)

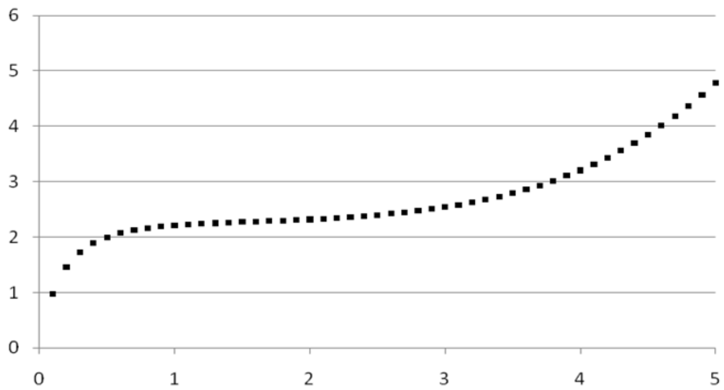

Figure 2. The activity signature plot for PKN Orlen (vertical axis – exponents p, horizontal axis – the values of $\hat{\beta}(p;2,h)$)

quadratic variance and realized variance was small, so we decide that microstructure effect for this frequency is negligible.

The Figure 2 represents the activity signature plot for PKN Orlen, but the graph is similar to the plots for liquid stock, future contract and currency prices. The plot resemble the case (17), when the process contains both jump and diffusion parts. The plot for index WIG, shown on the Figure 3, is typical for all indexes. In this case the process is continuous.

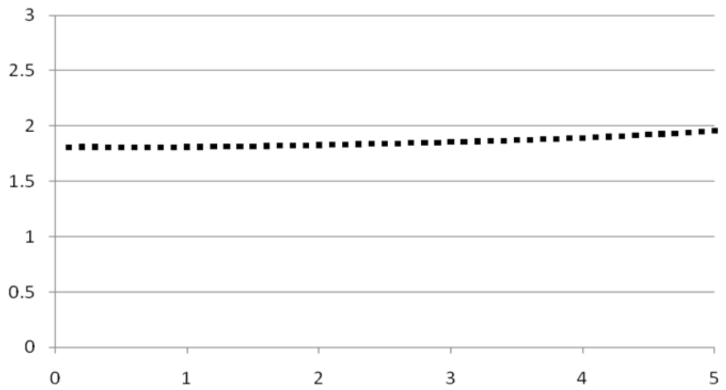

Figure 3. The activity signature plot for WIG (vertical axis – exponents p, horizontal axis – the values of $\hat{\beta}(p;2,h)$)

3. Estimating Blumenthal-Getoor Index Using Threshold Estimator

The method of activity signatures allows us to identify the type of process that underlines prices, but usually does not allow estimating the value of Blumenthal-Getoor index. To estimate the values of this index we use threshold estimator proposed by Aït-Sahalia and Jacod (2009). The estimator is given by the formula:

$$\hat{\beta}(\alpha,k,h)=\frac{\ln U(\alpha,h)-\ln U(\alpha k,h)}{\ln k}, \tag{18}$$

where $U(\alpha,h)$ is counting function which counts the exceedances of the threshold:

$$U(\alpha,h)=\sum_{i=1}^{N}1_{\{|x_i(h)|>\alpha h^{\omega}\}}, \tag{19}$$

with $\omega\in(0,1/2)$. Estimator (18) uses two time scales, as activity signature method, but the latter method is based on limit properties of power variation, while the threshold estimator is based on different exceedance rates in different time scales. It allows us to estimate Blumenthal-Getoor index in the case when the process contains diffusion (continuous) part. It also allows for testing the accuracy of the estimation. The asymptotic standard error of the estimator equals:

$$\frac{1}{\ln k}\sqrt{\frac{1}{U(\alpha k,h)}-\frac{1}{U(\alpha,h)}}. \tag{20}$$

We have calculated the estimator $\hat{\beta}(\alpha,k,h)$ for all instruments in the sample. As in the original work of Aït-Sahalia and Jacod (2009) we have taken $\omega = 1/5$ and $k = 2$. As for the threshold level α it was taken as seven times the estimated standard error of the continuous part of the process. The results are shown in Table 1.

Table 1. Estimators of Blumenthal-Getoor index

Instrument	$\hat{\beta}(\alpha,k,h)$	Std. dev.
AGO	0.7382	0.0087
BRE	1.0888	0.0074
KGH	1.5486	0.0052
PKN	1.5405	0.0085
WIG	2.6208	0.0285
WIG20	1.9684	0.0097
MWIG40	2.2870	0.0185
FWIG20	1.9280	0.0024
EUR	1.9835	0.0097
USD	2.0491	0.0069

The results are only partially consistent with the ones obtained with activity signature method in the previous section. For the less-liquid stocks (AGO and BRE) the estimators of β are significantly lower than for the other instruments. They are however greater than 0. In case of stock indexes (WIG, WIG20 and MWIG40) the estimators are high (close to 2 or even greater than 2), what is consistent with previous results, that trajectories of these instruments are continuous. Liquid stocks have β between 1 and 2. As for the currencies (EUR, USD) and futures contract (FWIG20), the obtained values are close to 2, what stands in contradiction with the results from the previous section. The last column in the Table 1 contains estimators' errors. However they were calculated with asymptotic formula (20) and it is dubious if they give the true errors of estimators, especially if some values are greater than 2.

4. Singularity Spectra and Jump Activity

The third method of estimating Blumenthal-Getoor index is based on singularity spectra of observed trajectories of prices. It is non-statistical methods. It is based on the fact that the trajectories of Lévy processes with different β almost surely (i.e. with probability 1) reveals different types of continuity. Let us first introduce the concept of singularity spectrum.

Take any function $f: R \to R$. We say that the function is a-Hölder continuous in the point t_0 if there exists a polynomial $P(t)$ of order lower then a such that in some neighborhood of t_0:

$$\left| f(t) - P(t) \right| \le K \left| t - t_0 \right|^a \tag{21}$$

for some $K > 0$. Let α be a number such that for $a < \alpha$ function f is a-Hölder continuous at t and for all $a > \alpha$ f is not a-Hölder continuous at t. The α is called local Hölder exponent of the function f at the point t and is denoted by $h_f(t)$. It is the measure of "regularity" of f in the neighborhood of t. The higher $h_f(t)$, the more regular the function is. For example the trajectories of Wiener motion almost surely have local Hölder exponent equal to 0.5 at each point. Let $\Omega_f(\alpha)$ be the set of all points in which the function f has local Hölder exponent equal to α:

$$\Omega_f(\alpha) = \{ t : h_f(t) = \alpha \}. \tag{22}$$

Singularity spectrum it is the mapping which for all α returns the Hausdorff dimension (see Mallat, 2003; Falconer, 2003) of the set $\Omega_f(\alpha)$:

$$D_f(\alpha) = \dim_{\mathrm{H}} \Omega_f(\alpha). \tag{23}$$

As was shown by Jaffard (1999) the Lévy processes with different Blumenthal-Getoor index almost surely have different singularity spectra. If the Lévy process L does not contain diffusion part, then:

$$\dim_H \Omega_L(\alpha) = \alpha\beta \quad \text{for } \alpha \le \frac{1}{\beta} \tag{24}$$

and $\Omega_L(\alpha) = \varnothing$ for $\alpha > 1/\beta$. The shape of singularity spectrum for such a process is show in the Figure 4. If the process L contains diffusion part, then:

$$\dim_H \Omega_L(\alpha) = \alpha\beta \quad \text{for } \alpha \le \frac{1}{2}, \tag{25}$$

$\dim_H \Omega_L(1/2) = 1$ and $\Omega_L(\alpha) = \varnothing$ for $\alpha > 1/2$. For the Wiener process (without jumps) $\dim_H \Omega_L(1/2) = 1$ and $\Omega_L(\alpha) = \varnothing$ for $\alpha \ne 1/2$.

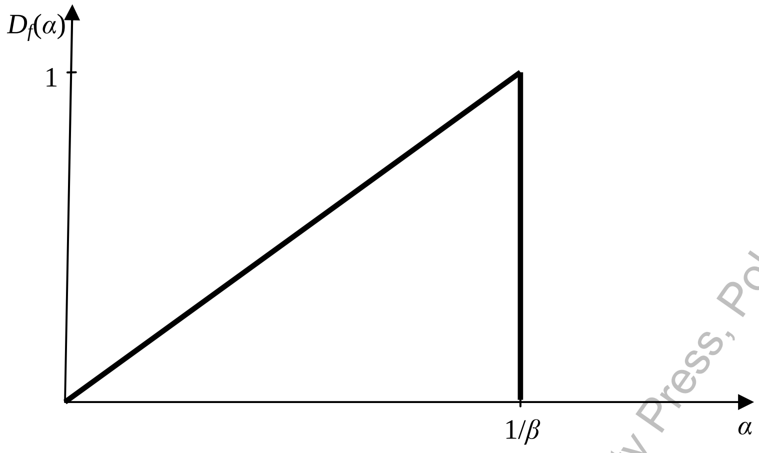

Figure 4. Singularity spectrum for Lévy process with Blumenthal-Getoor index β

To calculate singularity spectra from discretely sampled data one uses so called “multifractal formalism”, introduced by Frisch and Parsi (1985) and developed by Jaffard (1997a, 1997b). It can be shown that the function D_f^*, defined as:

$$D_f^*(q) = \liminf_{h\to 0} \frac{1}{\ln h} \ln \int \left| f(x+h) - f(x) \right|^q dx \qquad (26)$$

is the Legendre transform[4] of the singularity spectrum D, i.e.:

$$D_f^*(q) = \inf_{\alpha\in R} \left(q\alpha - D_f(\alpha) \right). \qquad (27)$$

If the function $D_f(\alpha)$ is convex (as it is generally assumed in the literature), then one can obtain D_f by performing Legendre transform on D_f^*. The function D_f^* can be estimated from sample moments. One has to calculate power variation (10) for multiple of sampling times h and then to use the regression:

$$\ln V(q,h) = C + D_f^*(q) \ln h\,. \qquad (28)$$

This method however is not stable numerically and some better methods were proposed. Most of them use wavelet transform. The review of them can be found in Turiel, Pérez-Vincente, Grazzini (2006) or Oświęcimek (2005). In our research we have used method based on “modulus maxima” of wavelet coefficients, implemented in Matlab package Fraclab. The method was pro-

[4] Lagendre transform is well-established method of convex analysis. See for example Rockefaller (1970), ch. 26.

posed by Jaffard (1997a) and very good description can be found in Mallat (2003, ch. 6).

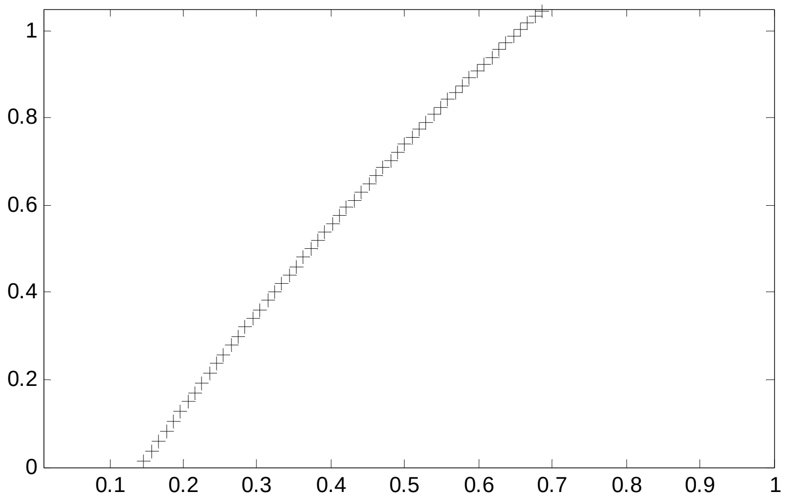

Figure 5. Estimated singularity spectrum for PKN Orlen

The Figure 5 shows the results for one stock (PKN Orlen). The estimated singularity spectra for all other instrument revealed very similar pattern, so we omit them. One should also stress that the results bears a great resemblance to the results of Cont and Tankov (2004) for the stock from US market. The graph does not star at the origin, which is inconsistent with typical shape depicted on the Figure 4. This however can be due to numerical inaccuracy. The main feature of the graph is the value at which the function reaches its peak, as it is the reciprocal of Blumenthal-Getoor index. For all instrument in the sample the peak values lie in the interval [0.6, 0.8], which means that the indexes β are between 1.2 and 1.8.

Conclusions

The estimation of the Blumenthal-Getoor index is a complicated problem. We have used three different methods and as one can see the results are in many cases inconsistent.

As for the less-liquid stocks, according to the activity signature method the price process is driven by compound Poisson process, while the two other methods reveal positive value of Blumenthal-Getoor index. This inconsistency may be due to the fact of low liquidity of these stocks. There were days when the prices did not change for several hours. The estimators we have used take advantage of limit properties of price changes as time scale tends to 0. It is thus dubious if they gave correct estimators for such illiquid stocks.

As for the liquid stocks all three estimators gave similar results. Moreover the results are consistent with similar results for other markets (β at the level about 1.5). According to the first two methods (activity signature and threshold estimator) the indexes are continuous processes, while the third method (singularity spectrum) revealed $\beta < 2$. Probably in fact the former result holds, as the method of singularity spectrum is the most prone to numerical errors.

The results for currencies and futures contract are ambiguous. This can be a result of active process of jumps with Blumenthal-Getoor index close to 2. As it was pointed out by Zhang (2007) the jump process is then hardly distinguishable from continuous diffusion.

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Aktywność skoków i spectrum osobliwości dla instrumentów z polskiego rynku finansowego

Z a r y s t r e ś c i. W artykule podejmujemy próbę oszacowania aktywności skoków w procesach cen kilku instrumentów z polskiego rynku finansowego. Jako miarę aktywności skoków przyjmujemy indeks β Blumenthala-Getoora dla procesów Lévy'ego. Pozwala nam to na rozróżnienie procesów charakteryzujących się rzadkimi i dużymi skokami i procesów o nieskończonej aktywności procesu skoków. Aktywność skoków szacujemy trzema różnymi metodami. Wykorzystujemy wykresy podpisu aktywności (*activity signature plots*) do zbadania typu procesu. Następnie korzystamy z estymatora Aït-Sahalii i Jacod, opartego na liczbie przekroczeń, do oszacowania wartości indeksu β. Wreszcie korzystamy ze spektrum ciągłości oraz z odpowiednich twierdzeń na temat przebiegu tej funkcji dla procesów Lévy'ego z różnymi wartościami indeksu β.

S ł o w a k l u c z o w e: wykładnicze modele Lévy'ego, indeks Blumenthala-Getoora, spektrum osobliwości.