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Decomposition of Risk and Quantile Risk Measures

1. Introduction

Portfolio managers led to the development of two things: first, correlation models, and second, analytical techniques that can handle asymmetrical risk. With correlation modeling, the subject has by and large converged on the conditional independence framework, allowing concrete statements to be made about default rate volatilities and the potential losses; this is now used both in risk management and in pricing. As regards portfolio analytics, the necessity for risk measures other than the normal mean-variance has been understood, and it is well established that expected shortfall (ES) is a good tool, as it is sensitive to tail risk, it is a coherent risk measure (Artzner *et al*, 1999, and Acerbi and Tasche, 2002), and it provides a close link to the notion of tranche payouts and hence to the CDO and securitisation world. We discuss the application of ES as quantitative measure of how much risk in the portfolio (model) comes from systematic risk and how much is residual. If we can assume that the risk is driven by a univariate risk factor A say, and that the A-conditional expected loss of the portfolio, $E[Y \mid A]$, is a monotonic function of this factor, and also that the portfolio is 'infinitely fine-grained', then the portfolio loss is a one-to-one transformation of the risk factor: $Y = E[Y \mid A]$. The Basel II framework chooses a simple prescription, thereby allowing the quantiles of $E[Y \mid A]$ to be easily computable. Now if we wish to incorporate the effects of unsystematic risk (finite portfolio; large or largish exposures) we can model the loss as $Y = X + U$, with $X = E[Y \mid A]$ and U denoting an independent Gaussian residual of variance σ^2. The difference between the upper P-quantiles of X and Y is given by the formula (Martin and Wilde, 2002), and references therein):

$$VaR_P(Y) \approx VaR_P(X) - \frac{1}{2f(x)} \frac{d}{dx} \sigma^2(x) f(x)\Big|_{x=VaR_P(X)} \tag{1}$$

where f is the density of X. Incidentally the shortfall[1] is:

$$ES^+{}_P(Y) \approx ES^+{}_P(X) - \frac{1}{2P} \sigma^2(x) f(x)\Big|_{x=VaR_P(X)} .$$

Notice that for the purposes of the GA formulas above we do not need A to be univariate; however, if it is not, then it is generally a lot more difficult to calculate the VAR of $E[Y|A]$. From now on, we will not make any assumptions about the distribution of A, or about the conditional distribution of Y on A; we thereby keep everything general.

Continuing from the above ideas, we start by writing:

$$Y = \mu_{Y|A} + (Y - \mu_{Y|A}),$$

with $\mu_{Y|A} = E[Y|A]$ denoting the conditional mean of Y given A (so it is a random variable). It is then natural to consider the following expression:

$$E[Y| Y > y] = E[\mu_{Y|A} | Y > y] + \text{remainder},$$

thereby splitting the ES into two parts.

In general, we will not know the distribution of $\mu_{Y|A}$ in 'closed form'. In practice this is not an issue because when calculations are done one is in effect coming up with a large number of scenarios for A and computing the conditional mean $E[Y|A]$ in each; the distribution of $\mu_{Y|A}$ is then approximated by the empirical distribution of the generated sample.

We identify the first term as the contribution of systematic risk to the portfolio ES (or the 'systematic part' for short). Incidentally, this arises as a natural consequence of the Fourier integral representation of shortfall, used in the saddle point approximation.

We explore the relationship between this and the well-known analysis of variance formula:

$$D^2(Y) = D^2(\mu_{Y|A}) + E(\sigma^2_{Y|A}) \tag{2}$$

in which the terms on the right-hand side tell us how much risk comes from the variation of the risk factor(s) $D^2(\mu_{Y|A})$ and how much comes from residual risk $E[\sigma^2_{Y|A}]$). Hence we identify the 'remainder' term in the above equation as the contribution to unsystematic risk. For a multivariate normal portfolio model, and for elliptical distributions (such as Student-t) the decompositions are essentially identical.

[1] The ES: S^+ or S^- is defined by $E[Y | Y > y]$ or $E[Y | Y < y]$ where Y is the portfolio loss (or value) and y is the VaR at the chosen tail probability. In the case of Y not having a continuous distribution.

2. Systematic decomposition of ES

We start with our basic definition

$$E[\mu_{Y|A} \mid Y > y]. \tag{3}$$

This can also be written as the mean plus the covariance of the conditional expected loss and tail probability:

$$\mu_Y + \frac{1}{P(Y > y)} \operatorname{cov}(\mu_{Y|A}, P(Y > y|A)), \tag{4}$$

with $\mu_Y = E[Y]$. (This is apparent when one expands the covariance as the expectation of the product minus the product of the expectations; the second term cancels the μ_Y.) There is a link with mean-variance theory: the 'mean plus some number (η) of standard deviations' risk measure is:

$$\mu_Y + \eta\sigma_Y = \mu_Y + \frac{\eta}{\sigma_Y} D^2(\mu_{Y|A}) + \frac{\eta}{\sigma_Y} E(\sigma^2_{Y|A}), \tag{5}$$

so the second term is the covariance of $\mu_{Y|A}$ with itself (multiply by η/σ_Y). Hence the expressions for systematic risk contribution have in common that they are the mean plus the covariance of the conditional expected loss with something reasonably natural (the conditional tail probability in (4), the conditional expected loss in (5)). As we shall see later, in the case when the joint distribution of asset returns in the portfolio is multivariate normal, the decompositions are in fact identical.

3. Multivariate normal model and elliptic model

We drew a comparison between the mean-variance and ES frameworks earlier (equations (4) and (5)). Pursuing this line a bit further, one might ask whether the ES and mean-variance decompositions the same result in any particular case. We start with the multivariate normal portfolio and risk factor, on which the ES and mean-variance measures are equivalent. Let the risk factor be $A \sim N(0, \Sigma)$ and write:

$$Y = \mu + k' A + U,$$

where k is the vector of the factor weights and $U \sim N(0, \sigma^2_U)$ is independent of A and represents the unsystematic risk. Then:

$$\mu_{Y|A} = \mu + k' A,$$

$$\sigma^2_{Y|A} = \sigma^2_U,$$

$$f_{Y|A}(y) = \frac{1}{\sigma_U} \phi\left(\frac{y - \mu - k'A}{\sigma_U}\right),$$

$$P[Y > y|A] = \Phi(\frac{y-\mu-k'A}{\sigma_U})$$

and so the systematic contribution to ES is:

$$\frac{1}{P^+}E[(\mu+k'A)\Phi(\frac{y-\mu-k'A}{\sigma_U})] = \frac{1}{P^+}[\mu\Phi(\frac{\mu-y}{\sigma_Y})+\frac{k'\sum k}{\sigma_Y}\Phi(\frac{\mu-y}{\sigma_Y})],$$

(as the integration over *A* can be done in closed form) while unsystematic contribution is:

$$\frac{1}{P^+}E[\sigma_U^2 f_{y|A}] = \frac{1}{P^+}\frac{\sigma_U^2}{\sigma_Y}\phi(\frac{\mu-y}{\sigma}),$$

where $\sigma_Y^2 = k'\Sigma k + \sigma_U^2$ is the unconditional variance of *Y*. Note that $P^+ = \Phi(\frac{\mu-y}{\sigma_Y})$ so the first term in the systematic part just μ, as expected.

To compare this with the mean-variance framework, let the mean – variance risk measure be mean plus ηstandard deviations, that is, $\mu_Y + \eta\sigma_Y$. Then the systematic and unsystematic parts are respectively:

$$\mu+\eta\frac{k'\sum k}{\sigma_Y} \text{ and } \eta\frac{\sigma_U^2}{\sigma_Y}.$$

Setting $\eta = \phi(\Phi^{-1}(P^+))/P^+$, a definition that depends only on the choice of tail probability rather than on the portfolio in question, makes the ES and mean-variance decompositions agree. The systematic part is proportional to the square of the correlations (that is, '*k* squared') and the unsystematic part is inversely proportional to the portfolio size (not inversely proportional to its square root). As η is independent of the portfolio mean and variance, the result extends automatically to elliptical distributions of finite variance, as the elliptical model is obtained from the normal model by making the variance random and then integrating it out (for example, for Student-*t* it is reciprocal-gamma distributed).

4. Empirical analysis

We take as a test an example of set of assets belong to portfolio WIG-Media from Warsaw Stock Exchange in period which cover this new index that means from 15.02.2005 up to day 28.05.2007. On this period rate of return of market benchmark, WIG was on level 0.00155 with variance equals 0.00013. On media sector, index WIG-Media rate of return had mean -0.00064 and variance 0.00166. We started from classical risk decomposition based on Sharp model (table 1). Ratio in this table means ratio of market risk to global risk.

Table 1. Results of risk decomposition based on Sharp model

	AGORA	INTERIA	MNI	MUZA	PWK	TVN	WSIP
mean	-0.00027	0.00376	0.00188	0.00250	0.00201	0.00056	0.00097
σ	0.02239	0.03071	0.04084	0.05021	0.05773	0.03953	0.01730
β	0.22159	0.06113	0.04949	0.05895	0.03187	0.06177	0.15072
systematic risk	0.00257	0.00137	0.00152	0.00241	0.00160	0.00357	0.00596
unsystematic risk	0.02141	0.03060	0.04074	0.05004	0.05768	0.03939	0.01699
ratio	0.02239	0.03071	0.04084	0.05021	0.05773	0.03953	0.01730

Source: Own calculation.

Next we took to our analysis quantile risk measures as was described early. This example, by contrast, shows that decomposition done by ES and mean-variance do not always agree. We not assume normal distribution and statistical tests give rejections of hypotheses of normality (figure 1 gave the graphical representations of tests). We have on modal distributions but not normal. Results of risk decomposition based on ES model we notice in table 2 and 3, were we collaborated with two different confidence level 0.05 and 0.01.

Table 2. Results of risk decomposition based on ES model (confidence level 0.05)

	AGORA	INTERIA	MNI	MUZA	PWK	TVN	WSIP
$VaR_{0,05}$	-0.0352	-0.0371	-0.0353	-0.0639	-0.0547	-0.0319	-0.0246
$CVaR_{0,05}$	-0.0511	-0.0548	-0.0751	-0.0880	-0.0970	-0.0713	-0.0364
β	2.0598	2.1586	10.3178	3.0259	10.177	11.4423	1.2646
systematic risk	0.0168	0.0176	0.0839	0.0246	0.0828	0.0930	0.0103
unsystematic risk	0.0061	0.0060	0.0819	0.0091	0.0832	0.1022	0.0025
ratio	0.7319	0.7462	0.5060	0.7306	0.4985	0.4765	0.8031

Source: Own calculation.

Table 3. Results of risk decomposition based on ES model (confidence level 0.01)

	AGORA	INTERIA	MNI	MUZA	PWK	TVN	WSIP
$VaR_{0,01}$	-0.0670	-0.0713	-0.0600	-0.0915	-0.0925	-0.0512	-0.0422
$CvaR_{0,01}$	-0.0821	-0.0856	-0.1865	-0.1264	-0.2069	-0.1923	-0.0533
β	1.5441	1.0348	24.0445	3.7723	22.8118	28.0666	1.0904
systematic risk	0.0123	0.0083	0.1919	0.0301	0.1821	0.2240	0.0087
unsystematic risk	0.0055	0.0073	0.0960	0.0141	0.1212	0.1338	0.0046
ratio	0.6909	0.5308	0.6666	0.6811	0.6005	0.6261	0.6520

Source: Own calculation.

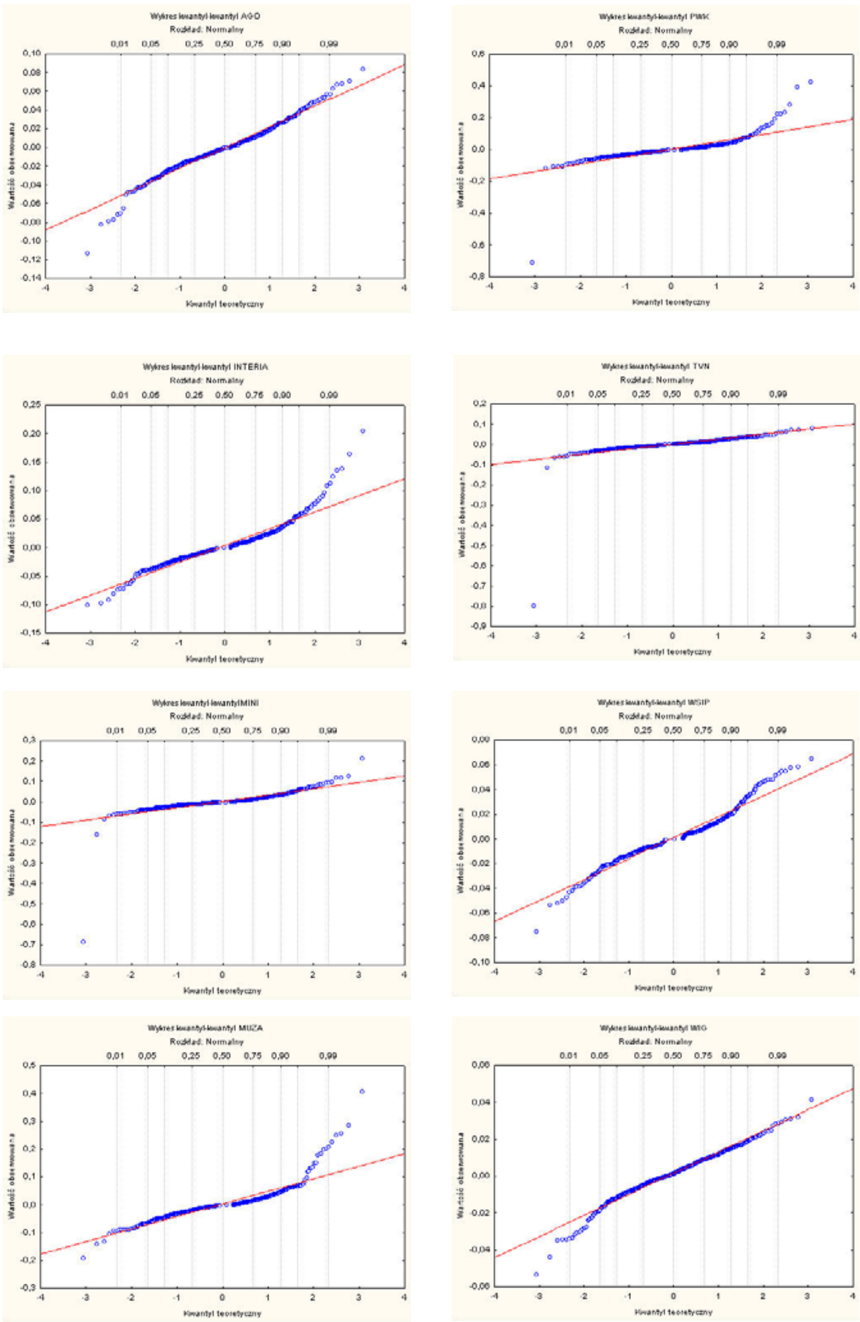

Figure 1. Quantile –quantile normality plot
Source: Own calculation.

We can observe that the level of systematic and unsystematic risk depend not only on confidence level but also on asymmetry on probability tail of return distributions. We can also observe how much changed beta value when we look close only on tail of return distributions.

An interpretation is unnecessary for the development of the theory. In one-factor risk models, A corresponds to the 'general state of the economy' say hole market. Complex portfolios require multi-factor models, and it is quite commons to use industrial or different sectors of market to represent the different risk factors. A question that naturally arises is whether portfolio risk can be split down by issuer and by 'systematic/unsystematic' at the same time and how it depend on type of return distributions.

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